PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Geometry of Dynamics

J. L. Synge

Phil. Trans. R. Soc. Lond. A 1927 **226**, 31-106
doi: 10.1098/rsta.1927.0002

II.—*On the Geometry of Dynamics.*

By Prof. J. L. Synge, *Fellow of Trinity College, Dublin.*

Communicated by Prof. A. W. Conway, *F.R.S.*

(Received January 18,—Read March 25, 1926.)

Contents.

 [Published, August 17, 1926.

PAGE

CHAPTER IV.

Study of the Manifold of Configurations with the Action Line-element.

$$ds^2 = 2(h - V)\,Tdt^2 = (h - V)\,a_{mn}dq^m dq^n = g_{mn}dq^m dq^n.$$

CHAPTER V.

Ignorability of Co-ordinates with the Kinematical Line-element.

CHAPTER VI.

Definitions of Stability and of Steady Motion.

CHAPTER VII.

Stability in the Kinematical Sense.

CHAPTER VIII.

Stability in the Kinematico-statical Sense.

INTRODUCTION.

To the pure mathematician of the present day the tensor calculus is a notation of differential geometry, of special utility in connection with multi-dimensional spaces; to the applied mathematician it is the backbone of the general theory of relativity. But when it is recognised that every problem in applied mathematics may be regarded as a geometrical problem (in the widest sense) and that the geometrical forms which many of these problems take are such that the tensor calculus can be directly applied, it is realised that the possibilities of this calculus in the field of applied mathematics can hardly be overestimated. It has a dual importance: first, by its help, known results may be exhibited in the most compact form; secondly, it enables the mathematician to exercise his most potent instrument of discovery, geometrical intuition.

In the present paper we are concerned with the development of general dynamical theory with the aid of the tensor calculus. In view of the present close association of the tensor theory with the theory of relativity, it should be clearly understood that this paper only attempts to deal with the classical or Newtonian dynamics of a system of particles or of rigid bodies. The subject is presented in a semi-geometrical aspect, and the reader should visualise the results in order to realise the close analogy between general dynamical theory and the dynamics of a particle. Mathematicians display a strange reluctance in summoning to their assistance the power of visualisation in multi-dimensional space. They forget that they have studied the geometry of three dimensions largely through the medium of a schematic representation on a two-dimensional sheet of paper. The same method is available in the case of any number of dimensions.

The reasoning of the present paper is essentially analytical, but the results are translated as far as possible into geometrical language; it is to stress the geometrical character of the theory that the title "Geometry of Dynamics" is adopted in preference to that more commonly used in kindred discussions, viz., "The General Problem of Dynamics."

In Chapter I. the various types of dynamical manifold are outlined. The further study of these manifolds with the aid of the tensor calculus is suggested as a fruitful field of research. They afford interesting extensions of familiar geometrical ideas. For example, the manifold of configurations of a top with fixed vertex is a three-dimensional manifold with a novel connectivity. Chapter II. is concerned with certain geometrical properties of Riemannian space. Chapters III. and IV. contain discussions of the manifold of configurations with the two important types of line-element. Attention is directed to the generalisation of BONNET'S Theorem (§ 3.3), the Principle of Least Curvature (§§ 3.6, 4.5) and the determinate form of LAGRANGE'S equations applicable to non-holonomic systems (§ 3.8). In Chapter V. there are given necessary and sufficient conditions that all the co-ordinates but one should be ignorable.

The rest of the paper deals with the question of the stability of states of motion. The discussion is based on the Lagrangian equations, which are well adapted for the tensorial method. In Chapter VI. there is given a geometrical definition of stability which yields three types of dynamical stability. The current definition of steady motion, so well suited to the Hamiltonian method, is replaced by definitions modelled to meet the requirements of the subsequent discussion of vibrations about a state of motion. The definitions adopted are, however, connected with the current definition. Chapters VII., VIII. and IX. contain analytical treatments of the problem of the stability of a state of motion (not necessarily steady). The curvature tensor makes its appearance in these chapters. On account of the rather heavy analysis connected with the case of N degrees of freedom, the important cases of two and three degrees of freedom receive preliminary treatment. The method employed is in a sense a generalisation of the method of moving axes. The generalised Frenet-Serret formulæ, due to BLASCHKE, play a fundamental part, the displacement from the undisturbed to the disturbed trajectory being resolved into components along the tangent and the normals to the trajectory. In Chapter IX. the theory is purely geometrical, the time being eliminated by the condition of fixed total energy. The results are simpler than those obtained in the preceding chapters, the question of stability being merely a problem in the geometry of geodesics in a Riemannian space. However, there is the compensating disadvantage that only disturbances which do not change the total energy are taken into consideration. Some simple applications of the tests are given.

CHAPTER I.

TYPES OF DYNAMICAL MANIFOLD.

§ 1.1. *The manifold of configurations.*

Each point of the manifold of configurations represents a configuration of the system. If the dynamical system has N generalised co-ordinates q^r ($r = 1, 2, \ldots, N$), this manifold is N-dimensional with co-ordinates q^r. When a point of the manifold is given, the positions of all the particles forming the system are also given. This is the dynamical manifold most commonly considered.

There are two types of line-element of particular importance. We shall assume throughout that the kinetic energy T is homogeneous and quadratic in the generalised velocities and does not contain the time explicitly. Let us write

$$ds^2 = 2T dt^2 = a_{mn} dq^m dq^n. \tag{1.11}$$

The force system is not involved in this definition of ds and for that reason we shall call (1.11) the *kinematical line-element* in the manifold of configurations.† It is not assumed that the system is holonomic.

The second type of line-element is

$$ds^2 = 2(h - V)\,Tdt^2 = (h - V)\,a_{mn}dq^m dq^n, \tag{1.12}$$

where h is a constant and V is the potential energy. This type of line-element is obviously only available for the discussion of conservative systems, and, indeed, is only of importance for the discussion of motions having a total energy h. In that case the natural motion takes place along a geodesic of the manifold, a familiar deduction from the Principle of Least Action.‡ For that reason we shall call (1.12) the *action line-element* in the manifold of configurations.

§ 1.2. *The manifold of configurations and time.*

Each point of the manifold of configurations and time represents a configuration and an instant. The manifold is of $(N + 1)$ dimensions, the co-ordinates of a point being q^r $(r = 1, 2, \ldots, N)$ and t. When a point of the manifold is given, the positions of all the particles forming the system and the time are also given.

It is more difficult to pick out a line-element in this manifold, for although many are available they do not appear to be of much interest. The following line-element suggests itself as a possible basis for the geometry of the manifold of configurations and time :—

$$ds^2 = 2Ldt^2 = a_{mn}dq^m dq^n - 2Vdt^2. \tag{1.21}$$

This bears a certain resemblance to the line-element in a stationary gravitational field in the general theory of relativity.

But there is another form

$$ds = 2L\,dt = \frac{a_{mn}dq^m dq^n}{dt} - 2V\,dt. \tag{1.22}$$

† Certain aspects of the geometrico-dynamical theory of the manifold of configurations with this line-element have been discussed by M. Lévy, *Comptes rendus*, 86 (1878), 875 ; G. Darboux, *Théorie des Surfaces*, Pt. 2 (1915), 516 ; E. Kasner, *Trans. Amer. Math. Soc.*, 10 (1909), 201 ; J. Lipka, *Trans. Amer. Math. Soc.*, 13 (1912), 77, *Proc. Amer. Acad. of Arts and Sciences*, 55 (1920), 285, *Bull. Amer. Math. Soc.*, 27 (1920), 71, *Journal of Math. and Phys.*, *Massachusetts Inst. of Technology*, 1 (1921), 21 ; L. M. Kells, *Amer. Journal of Math.*, 46 (1924), 258. None of these writers appear to have made use of the tensorial notation. The only work with which I am acquainted in which this notation has been applied to classical dynamics is the well-known memoir of Ricci and Levi-Civita, *Math. Ann.*, 54 (1900), 178–190 (*cf.* J. E. Wright, "Invariants of Quadratic Differential Forms," *Cambridge Tracts*, No. 9, 80). H. Hertz, *Principles of Mechanics* (translated by Jones and Walley, 1899), considered the manifold of $3N$ dimensions corresponding to a system of N particles, using a line-element essentially the same as the kinematical line-element. The line-element being a sum of squares of differentials with constant coefficients, Hertz was able to proceed without the tensorial notation.

‡ *Cf.* Appell, *Mécanique rationnelle*, 2 (1911), 436.

For this line-element the natural trajectories are geodesics by virtue of Hamilton's Principle. It is possible that the development of a geometrico-dynamical theory based on this line-element would form an interesting subject for research. However, since ds is not the square root of a homogeneous quadratic form, the geometry is not Riemannian and there is a certain difficulty in the definition of angle in such a manifold.†

§ 1.3. *The manifold of states.*

Each point of the manifold of states represents a state of the system. If p_r ($r = 1, 2, \ldots, N$) are the generalised components of momentum ($p_r = \partial T/\partial \dot{q}^r$), the co-ordinates of a point of the manifold are q^r, p_r ($r = 1, 2, \ldots N$); the manifold is therefore of $2N$ dimensions. When a point of the manifold is given, the positions and velocities of all the particles forming the sytem are also given.‡ This and the following manifolds appear to be non-metrical; their most interesting properties are associated with the integral-invariants.

§ 1.4. *The manifold of states and time.*

Each point of the manifold of states and time represents a state and an instant. The manifold is therefore of $(2N + 1)$ dimensions, the co-ordinates of a point being q^r, p_r ($r = 1, 2, \ldots, N$) and t. When a point of the manifold is given, the positions and velocities of all the particles forming the system and the time are also given.§

§ 1.5. *Note on the scope of the paper.*

The present paper treats only of the manifold of configurations (1.1), results being developed both for the kinematical line-element (1.11) and for the action line-element (1.12). As the theories connected with these two line-elements run more or less in parallel, the symbol (K) is placed after the number of every theorem in the enunciation of which the kinematical line-element is implied, with a similar use of the symbol (A) where the action line-element is understood. Where a theorem is stated in such a way that neither of these line-elements is necessarily involved, as in the cases of theorems of Riemannian geometry and of theorems of a purely dynamical significance, no such symbol is employed.

To avoid overloading the symbols with indices, I have thought it proper frequently to use the same symbol in two different but similar senses, the sense to be understood being sufficiently evident from the section or chapter in which it occurs. In any given section (other than those containing matter of a purely geometrical import) we have

† The geometry of this more general type of metrical manifold has been developed to a certain extent; *cf.* P. Finsler, "Ueber Kurven und Flächen in allgemeinen Räumen," *Dissertation, Göttingen* (1918); J. L. Synge, "A Generalisation of the Riemannian Line-Element," *Trans. Amer. Math. Soc.*, 27 (1925), 61; J. H. Taylor, "A Generalisation of Levi-Civita's Parallelism and the Frenet Formulas," *ibid.*, 246.

‡ For the use of the manifold of states in connection with statistical mechanics, see J. H. Jeans, *The Dynamical Theory of Gases* (1925), 69.

§ This is the manifold considered by Cartan, *Leçons sur les invariants intégraux* (1922).

under consideration a definite line-element, either kinematical or action, and such quantities as Christoffel symbols, curvature tensors, etc., are to be calculated with respect to the fundamental tensor belonging to the line-element under consideration. In any case where it is necessary to introduce symbols which are to be calculated for a line-element other than that to which the section is devoted, explicit statements make matters clear. We shall adopt g_{mn} as standard notation for the fundamental tensor of the line-element under consideration, so that when we are thinking of the kinematical line-element we have

$$g_{mn} = a_{mn}, \tag{1.51}$$

and when we are thinking of the action line-element,

$$g_{mn} = (h - V)\, a_{mn}, \tag{1.52}$$

where uniformly we write

$$2T = a_{mn} \frac{dq^m}{dt} \frac{dq^n}{dt}, \tag{1.53}$$

q^r being the co-ordinate system.

The force system is in all cases supposed to be independent of the time and of the velocities, so that the generalised forces are functions of position in the manifold of configurations.

It is important to distinguish between the words *curve* and *trajectory*. A curve is a purely geometrical concept in the manifold and consists of a one-dimensional continuum of points. A trajectory is a curve along which the co-ordinates are given as functions of the time. A *natural* trajectory corresponds to a motion under a given force system according to the laws of dynamics.

CHAPTER II.

Notation and Geometrical Preliminaries.

§ 2.1. *Conventions for summation ; magnitude of a vector ; angle.*

As stated in § 1.5, this paper deals with two different line-elements in the manifold of configurations. It seems therefore desirable to preface the geometrico-dynamical developments of later chapters with some definitions and theorems couched in purely geometrical language. Of much of the substance of this chapter it may be said that it is already known in some form or other. It is necessary, however, for clarity and uniformity in notation to give a list of formulæ for later use.

The manifold under consideration being of N dimensions, the common convention of summation with respect to indices repeated in a product is adopted, except when the indices are capital letters ; the range of summation is from 1 to N for small italic indices and from 1 to $(N - 1)$ for small Greek indices. Repeated capital indices imply no summation, unless such is indicated by the sign Σ. Small italic indices unrepeated imply a range of values from 1 to N, small Greek indices from 1 to $(N-1)$, while unrepeated capital indices imply no range of values, except where such is specifically indicated.

Indices which do not imply tensorial character are generally enclosed in round brackets, except in the case of numerical indices denoting powers. To avoid confusion between numerical tensorial indices and indices denoting powers, the former are printed in italics, *e.g.*, q^{2} means the second component of a contravariant vector, q^2 means "q squared."

If X^r is any contravariant vector, its *magnitude* is X where

(2.11) $$X^2 = g_{mn}X^mX^n, \qquad X \geqslant 0.$$

A *unit vector* is one whose magnitude is unity.

The angle θ between two contravariant vectors X^r and Y^r is given by

(2.12) $$\cos\theta = \frac{g_{mn}X^mY^n}{(g_{rs}X^rX^s \, . \, g_{tu}Y^tY^u)^{1/2}}.$$

§ 2.2. *The contravariant space derivative and the contravariant time-flux.*

We shall denote derivatives with respect to the arc s of a curve by an accent (*e.g.*, $q^{r'}$) and derivatives with respect to the time t by a superposed point (*e.g.*, $\dot{q}^r$).

If X^r is a contravariant vector given along a curve, we shall write

(2.21) $$\bar{X}^r = X^{r'} + \left\{ {mn \atop r} \right\} X^m q^{n'},$$

and call $\bar{X}^r$ the *contravariant space derivative* of X^r along the curve. Similarly, if X^r is given as a function of t along a trajectory, we shall write

(2.22) $$\hat{X}^r = \dot{X}^r + \left\{ {mn \atop r} \right\} X^m \dot{q}^n,$$

and call $\hat{X}^r$ the *contravariant time-flux* of X^r along the trajectory. It is well known that $\bar{X}^r$ and $\hat{X}^r$ are contravariant vectors.†

§ 2.3. *Relative curvature ; first curvature.*

Lipka‡ has given a very simple descriptive definition of the relative curvature of two curves. Let C and C^* be two curves touching one another at a point O. Let P and P^* be points on C and C^* respectively such that $OP = OP^* = s$ (say). Let $PP^* = \sigma$. Then the relative curvature of C and C^* is defined to be

(2.31) $$\kappa\,(C, C^*) = \lim_{s \to 0} 2\sigma/s^2.$$

† *Cf.* BIANCHI, *Lezioni di Geometria Differenziale*, 2_2 (1924), 790.

‡ *Bull. Amer. Math. Soc.*, 29 (1923), 345.

When the curve C^* is a geodesic, this definition gives us the curvature (absolute curvature) of C.† If we write

(2.32) $$q^{r'} = \lambda^r, \quad \kappa^r = \bar{\lambda}^r = q^{r''} + \begin{Bmatrix} mn \\ r \end{Bmatrix} q^{m'} q^{n'},$$

the curvature κ of any curve is the magnitude of the vector κ^r, so that

(2.33) $$\kappa^2 = g_{mn} \kappa^m \kappa^n.$$

The relative curvature of two curves C and C^* is given analytically by

(2.34) $$[\kappa (C, C^*)]^2 = g_{mn} (\kappa^m - \kappa^{*m}) (\kappa^n - \kappa^{*n}),$$

the unasterisked quantities being calculated for C, the asterisked for C^*.

The vector κ^r defines the first or principal normal of a curve, and thus, if ν^r denotes the unit vector in the direction of the principal normal,

(2.35) $$\kappa^r = \kappa \nu^r.$$

The curvature of a trajectory is defined to be the curvature of its curve.

§ 2.4. *Co-planar vectors and components.*

If X^r and Y^r are two vectors at a point, we shall say that the vector Z^r is *co-planar* with them if A and B exist so that

(2.41) $$Z^r = AX^r + BY^r.$$

Further, if X^r and Y^r are unit vectors, we shall call A and B the *components* of Z^r in the directions of X^r and Y^r respectively.

The same idea is available in the more general case. If $X^r_{(1)}, X^r_{(2)}, \ldots, X^r_{(M)}$ are M contravariant vectors at a point and if $A^{(1)}, A^{(2)}, \ldots, A^{(M)}$ exist so that

(2.42) $$Y^r = A^{(1)} X^r_{(1)} + A^{(2)} X^r_{(2)} + \ldots + A^{(M)} X^r_{(M)},$$

then we shall say that Y^r is coplanar with $X^r_{(1)}, X^r_{(2)}, \ldots, X^r_{(M)}$, and, if these latter are all unit vectors, then $A^{(1)}, A^{(2)}, \ldots, A^{(M)}$ are the components of Y^r in the directions of $X^r_{(1)}, X^r_{(2)}, \ldots, X^r_{(M)}$.

§ 2.5. *A special co-ordinate system; curvature of a surface geodesic.*

We shall use the word *surface* to denote a manifold of $(N - 1)$ dimensions immersed in the fundamental manifold of N dimensions. The properties of a surface are often expressible in simple form by the use of a special co-ordinate system. It is well known that the congruence of geodesics normal to a surface is a normal congruence and that

† *Cf.* Bianchi, *loc. cit.*, 455. The curvature here considered is the first or principal curvature. Curvatures of higher orders are discussed in § 2.7.

any two of the normal surfaces give equal intercepts on all the geodesics. Now if $q^1, q^2, \ldots, q^{N-1}$ be a co-ordinate system selected arbitrarily on the given surface, and if the normal geodesics be the parametric lines of q^N, q^N being the distance from the given surface measured along these geodesics, we have a system of co-ordinates† for which

$$ds^2 = g_{\mu\nu}\, dq^\mu\, dq^\nu + (dq^N)^2. \tag{2.51}$$

We shall call such a system of co-ordinates "geodesic orthogonal trajectory with respect to q^N" or briefly *G.O.T.* (q^N) since the parametric lines of q^N are geodesics and are the orthogonal trajectories of the surfaces $q^N =$ constant.‡

A family of *parallel* surfaces is defined by the property that any pair of the surfaces gives equal intercepts on all the orthogonal trajectories. It is easily seen that the orthogonal trajectories must be geodesics, and therefore, given a family of parallel surfaces, it is always possible to choose a *G.O.T.* (q^N) co-ordinate system such that the equations of the surfaces of the family are $q^N =$ constant.

A *G.O.T.* (q^N) co-ordinate system is characterised by the equations

$$g_{N\rho} = 0, \qquad g_{NN} = 1. \tag{2.52}$$

A simple expression for the curvature of a surface geodesic can be found when a *G.O.T.* (q^N) co-ordinate system is employed, the equation of the surface being $q^N =$ constant.§ The equations of a surface geodesic are

$$g_{\rho\mu} q^{\mu''} + \begin{bmatrix} \mu\nu \\ \rho \end{bmatrix} q^{\mu'} q^{\nu'} = 0, \tag{2.53}$$

and the components of curvature (relative to the fundamental manifold of N dimensions) are

$$\kappa_r = g_{rm} q^{m''} + \begin{bmatrix} mn \\ r \end{bmatrix} q^{m'} q^{n'}, \tag{2.54}$$

or, since the curve is contained in the surface $q^N =$ constant,

$$\kappa_r = g_{r\mu} q^{\mu''} + \begin{bmatrix} \mu\nu \\ r \end{bmatrix} q^{\mu'} q^{\nu'}. \tag{2.55}$$

Hence, using (2.53) and (2.52), we have

$$\begin{cases} \kappa_\rho = 0, \\ \kappa_N = \begin{bmatrix} \mu\nu \\ N \end{bmatrix} q^{\mu'} q^{\nu'} = -\tfrac{1}{2} \dfrac{\partial g_{\mu\nu}}{\partial q^N} q^{\mu'} q^{\nu'}, \end{cases} \tag{2.56}$$

so that, since $g^{NN} = 1$,

$$\kappa = -\tfrac{1}{2} \frac{\partial g_{\mu\nu}}{\partial q^N} q^{\mu'} q^{\nu'}. \tag{2.57}$$

† *Cf.* BIANCHI, *loc. cit.*, 423, 450; BIANCHI calls (2.51) "forma geodetica del ds^2".

‡ *Cf. Proc. National Academy of Sciences*, 8 (1922), 200.

§ *Cf.* BIANCHI, *loc. cit.*, 457; the present method is more direct.

The minus sign in this expression is adopted in order that the formula

(2.571) $$\kappa^r = \kappa\lambda^r$$

may be true, λ^r being the contravariant unit vector normal to the surface drawn in the direction of q^N increasing.

The above result does not supply us directly with a method by which, given the equation of a surface for an arbitrary co-ordinate system, we can determine the curvature of a surface geodesic in any assigned direction. We do not propose to answer that question in its general form, but will deal with a special case which is of importance later.

Let us suppose that for the general co-ordinate system q^r with fundamental tensor g_{mn} we are given a family of parallel surfaces whose equations are

(2.58) $$F(q^1, q^2, \ldots, q^N) = \text{constant}.$$

Let us choose a *G.O.T.* (p^N) co-ordinate system p^r for which the given parallel surfaces are p^N = constant, and let the fundamental tensor for this co-ordinate system be f_{mn}, where, as in (2.52),

(2.581) $$f_{N\rho} = 0, \quad f_{NN} = 1.$$

Let us distinguish Christoffel symbols for the p^r co-ordinate system by the subscript (p). Let the components of the unit vector normal to the system (2.58) be denoted by λ^r in the q^r co-ordinate system and by μ^r in the p^r co-ordinate system, so that

(2.582) $$\mu^\rho = 0, \quad \mu^N = 1; \qquad \mu_\rho = 0, \quad \mu_N = 1.$$

Consider the expression

(2.583) $$E = -\lambda_{rs} q^{r\prime} q^{s\prime},$$

where $q^{r\prime}$ is in any direction lying in the surface (2.58) through the point, and λ_{rs} is the covariant derivative of λ_r. It is clearly invariant, and therefore we have

(2.584) $$E = -\mu_{rs} p^{r\prime} p^{s\prime} = -\mu_{\rho\sigma} p^{\rho\prime} p^{\sigma\prime}.$$

But

(2.585) $$\mu_{\rho\sigma} = \frac{\partial \mu_\rho}{\partial p^\sigma} - \begin{Bmatrix} \rho\sigma \\ t \end{Bmatrix}_{(p)} \mu_t,$$

and thus, by (2.582),

(2.586) $$\mu_{\rho\sigma} = -\begin{Bmatrix} \rho\sigma \\ N \end{Bmatrix}_{(p)} = \tfrac{1}{2}\frac{\partial f_{\rho\sigma}}{\partial p^N}.$$

But by (2.57) the curvature of a surface geodesic is given by

(2.587) $$\kappa = -\tfrac{1}{2}\frac{\partial f_{\rho\sigma}}{\partial p^N} p^{\rho\prime} p^{\sigma\prime},$$

and thus, by (2.584) and (2.586),

(2.588) $$E = \kappa.$$

Hence we have the result:

THEOREM I :—*The curvature of a surface geodesic of one of a family of parallel surfaces is*

$$-\lambda_{rs}\, q^{r'} q^{s'},$$

where λ^r *is the unit vector having at every point a direction normal to the surface of the family through the point and* $q^{r'}$ *is the unit vector tangent to the surface geodesic.*

Now the unit vector having at every point a direction normal to the surface of the family (2.58) through the point is

$$\lambda^r = F^r/(g_{mn}F^m F^n)^{1/2}, \qquad (2.59)$$

where

$$F^r = g^{rm}F_m, \quad F_m = \partial F/\partial q^m. \qquad (2.591)$$

Thus we have

$$\lambda_r = F_r/(g^{mn}F_m F_n)^{1/2}, \qquad (2.592)$$

and by covariant differentiation

$$\lambda_{rs} = F_{rs}/(g^{mn}F_m F_n)^{1/2} - F_r \,.\, g^{tu}F_{ts}F_u/(g^{mn}F_m F_n)^{3/2}, \qquad (2.593)$$

and hence the following result :

THEOREM II :—*The curvature* κ *of a surface geodesic of a member of a family of parallel surfaces*

$$F(q^1, q^2, \ldots, q^N) = \text{constant}$$

is given by

$$\kappa\,(g^{mn}F_m F_n)^{3/2} = -\,(F_{rs}\,.\,g^{mn}F_m F_n - F_r\,.\,g^{mn}F_{ms}F_n)\, q^{r'} q^{s'}, \qquad (2.594)$$

where

$$F_r = \partial F/\partial q^r, \qquad F_{rs} = \partial F_r\Big/\partial q^s - \begin{Bmatrix} rs \\ u \end{Bmatrix} F_u, \qquad (2.595)$$

and $q^{r'}$ *is the unit vector tangent to the surface geodesic.*

§ 2.6. *Conditions that the orthogonal trajectories of a family of surfaces should be geodesics.*

We shall now find necessary and sufficient conditions that the orthogonal trajectories of a family of surfaces should be geodesics,† the equations of the family being

$$F(q^1, q^2, \ldots, q^N) = \text{constant}. \qquad (2.61)$$

If we are given a congruence (normal or not) defined by the equations

$$\frac{dq^1}{\lambda^1} = \frac{dq^2}{\lambda^2} = \ldots = \frac{dq^N}{\lambda^N}, \qquad (2.62)$$

† RICCI and LEVI-CIVITA, *loc. cit.*, 154, have given conditions that a congruence of an " ennuple " should be geodesic. However, it is hoped that the method of the present paper will be found more direct. *Cf.* also J. E. WRIGHT, " Invariants of Quadratic Differential Forms," 71.

λ^r being the unit vector having at every point the direction of the curve of the congruence passing through the point, these equations may be written

(2.63) $$q^{r'} = \lambda^r,$$

and hence, by differentiation with respect to the arc of the curve of the congruence,

(2.64) $$\kappa^r = \bar{\lambda}^r = \lambda^r_s q^{s'} = \lambda^r_s \lambda^s,$$

or in covariant form,

(2.65) $$\kappa_r = \lambda_{rs}\lambda^s,$$

where λ_{rs} is the covariant derivative of λ_r. Thus, since the vanishing of all the components of κ_r is the characteristic of a geodesic, we have the result:

THEOREM III :—*In order that the curves of a congruence may be geodesics, it is necessary and sufficient that*

(2.66) $$\lambda_{rs}\lambda^s = 0,$$

where λ^r is the unit vector everywhere co-directional with the congruence.

It is important to note that from the mere fact that λ^r is a unit vector we have

(2.661) $$\frac{\partial}{\partial q^s}(g^{nr}\lambda_r\lambda_n) = 0,$$

and therefore

(2.662) $$\lambda_{rs}\lambda^r = 0.$$

Now the orthogonal trajectories of the family of surfaces (2.61) are defined by the equations

(2.67) $$\frac{dq^1}{F^1} = \frac{dq^2}{F^2} = \ldots = \frac{dq^N}{F^N},$$

where

(2.671) $$F^r = g^{rm}F_m, \qquad F_m = \partial F/\partial q^m.$$

Thus the unit vector λ^r having everywhere the direction of the trajectory is

(2.672) $$\lambda^r = F^r/(g^{mn}F_mF_n)^{1/2}.$$

Thus, applying (2.593), we obtain

(2.673) $$(g^{mn}F_mF_n)^2\,\lambda_{rs}\lambda^s = (F_{rs}\,.\,g^{mn}F_mF_n - F_r\,.\,g^{mn}F_{ms}F_n)\,F^s.$$

The curves in question therefore are geodesics if and only if

(2.674) $$F_{rs}F^s\,.\,g^{mn}F_mF_n - F_r\,.\,g^{mn}F_{ms}F^sF_n = 0.$$

But this may be written

(2.675) $$\frac{F_{rm}F^m}{F_r} = \frac{g^{mn}F_{mt}F^tF_n}{g^{mn}F_mF_n},$$

it being remembered that summation with respect to a repeated index applies only to a product—not to a quotient. Thus the conditions

(2.676) $$\frac{F_{rm}F^m}{F_r} = \frac{F_{sm}F^m}{F_s}$$

are necessary. It is not difficult to prove that they are also sufficient, for if they are satisfied each of the equal fractions is equal to

(2.677) $$\frac{g^{ns}F_n F_{sm} F^m}{g^{ns}F_n F_s}$$

which is the right-hand side of (2.675). Hence we have the result:

THEOREM IV:—*In order that the orthogonal trajectories of a family of surfaces*

$$F(q^1, q^2, \ldots, q^N) = \text{constant}$$

should be geodesics, it is necessary and sufficient that

(2.68) $$\frac{g^{mn}F_{rm}F_n}{F_r} = \frac{g^{mn}F_{sm}F_n}{F_s},$$

where

(2.681) $$F_r = \partial F/\partial q^r, \qquad F_{rs} = \partial F_r/\partial q^s - \begin{Bmatrix} rs \\ u \end{Bmatrix} F_u.$$

§ 2.7. *The system of normals and curvatures for a curve.*

BLASCHKE† has developed a system of normals and curvatures for a curve in Riemannian space of N dimensions with a positive definite line-element. I have developed similar results‡ in the case where the line-element is not necessarily positive definite. The notation of this latter paper is more compact for the purposes of the present paper than that of BLASCHKE and I shall adopt it here with slight modifications, writing $\kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$ for the first, second, ..., $(N-1)$th curvatures, $\lambda^r_{(1)}, \lambda^r_{(2)}, \ldots, \lambda^r_{(N-1)}$ for the unit vectors in the directions of the first, second, ..., $(N-1)$th normals and $\lambda^r_{(0)}$ for the tangent unit vector. Thus $\kappa_{(1)}$ is the κ considered in § 2.3, $\lambda^r_{(1)}$ is ν^r and the first normal is the principal normal. The various normals and curvatures are connected by the generalised Frenet-Serret formulæ

(2.71) $$\begin{cases} \bar{\lambda}^r_{(0)} = \kappa_{(1)}\lambda^r_{(1)}, \\ \bar{\lambda}^r_{(1)} = \kappa_{(2)}\lambda^r_{(2)} - \kappa_{(1)}\lambda^r_{(0)}, \\ \bar{\lambda}^r_{(2)} = \kappa_{(3)}\lambda^r_{(3)} - \kappa_{(2)}\lambda^r_{(1)}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \bar{\lambda}^r_{(N-2)} = \kappa_{(N-1)}\lambda^r_{(N-1)} - \kappa_{(N-2)}\lambda^r_{(N-3)}, \\ \bar{\lambda}^r_{(N-1)} = \qquad\qquad - \kappa_{(N-1)}\lambda^r_{(N-2)}. \end{cases}$$

† *Math. Zeitschrift*, 6 (1920), 94.

‡ *Proc. International Mathematical Congress*, Toronto (1924).

These equations serve to define the curvatures and normals. As cases of particular importance, we may note that in two dimensions (with the notation of § 2.3)

(2.711) $$\bar{\lambda}^r = \kappa \nu^r, \qquad \bar{\nu}^r = -\kappa \lambda^r,$$

and in three dimensions (where to avoid the clumsy subscripts we put ρ^r for $\lambda^r_{(2)}$ and σ for $\kappa_{(2)}$)

(2.712) $$\bar{\lambda}^r = \kappa \nu^r, \qquad \bar{\nu}^r = \sigma \rho^r - \kappa \lambda^r, \qquad \bar{\rho}^r = -\sigma \nu^r.$$

If it should happen that the Mth curvature is zero, the Mth normal and all normals of higher order become indeterminate. We can, however, still use (2.71) by putting the Mth curvature and all curvatures of higher order equal to zero, $\lambda^r_{(M)}$, $\lambda^r_{(M+1)}$, ..., $\lambda^r_{(N-1)}$ being mutually perpendicular unit vectors normal to the curve, perpendicular to the $(M-1)$ existing normals and undergoing parallel propagation along the curve.

Thus we may always speak of " the $(N-1)$ normals of a curve," even if the curve has vanishing curvatures, recognising, however, in this case that some of the normals are to a certain extent arbitrarily selected. If only one of the curvatures $(\kappa_{(N-1)})$ vanishes, the $(N-1)$th normal is then uniquely defined (except with respect to sense) by the condition of being perpendicular to the $(N-2)$ existing normals and to the tangent. Thus, in Euclidean space of three dimensions, a curve of vanishing second curvature (torsion) has a unique second normal (bi-normal), which is propagated parallelly along the curve.

CHAPTER III.

Study of the Manifold of Configurations with the Kinematical Line-element.

$$ds^2 = 2T dt^2 = a_{mn}\, dq^m\, dq^n.$$

§ 3.1. *Kinematics.*

Before introducing the force system we shall proceed with some purely kinematical considerations, bearing in mind as an obvious source of suggestion for nomenclature the analogy between the motion of a point of the manifold and the motion of a particle in the Euclidean space of three dimensions.

We shall call the vector

(3.11) $$v^r = \dot{q}^r$$

the *velocity vector,* the magnitude of the velocity being

(3.111) $$v = (a_{mn} \dot{q}^m \dot{q}^n)^{1/2} = \dot{s} = (2T)^{1/2}.$$

Defining the *acceleration vector* f^r as the contravariant time-flux of the velocity vector, we have

(3.12) $$f^r = \bar{v}^r = \dot{v}^r + \left\{ {mn \atop r} \right\} v^m v^n = \ddot{q}^r + \left\{ {mn \atop r} \right\} \dot{q}^m \dot{q}^n,$$

the magnitude of the acceleration being

(3.121) $$f = (a_{mn} f^m f^n)^{1/2}.$$

Transforming the parameter from t to s, we find

(3.122) $$f^r = \ddot{s} q^{r'} + \dot{s}^2 \left(q^{r''} + \begin{Bmatrix} mn \\ r \end{Bmatrix} q^{m'} q^{n'}\right).$$

By (2.32) and (2.35) this may be written

(3.123) $$f^r = \ddot{s}\lambda^r + \kappa \dot{s}^2 \nu^r = \dot{v}\lambda^r + \kappa v^2 \nu^r,$$

where λ^r is the unit vector tangent to the trajectory, κ is the curvature and ν^r is the unit vector in the direction of the principal normal. Thus we have the result :

THEOREM V (K) :—*In any trajectory the acceleration vector is co-planar with the tangent and principal normal of the trajectory.*

Since λ^r and ν^r are unit vectors, their coefficients in (3.123) are the components of the acceleration vector in the directions of the tangent and principal normal respectively. But

(3.13) $$\dot{s}^2 = 2T,$$

and hence

(3.131) $$\ddot{s} = \dot{T}(2T)^{-\frac{1}{2}} = T'.$$

Thus (3.123) can also be written

(3.14) $$f^r = T'\lambda^r + 2\kappa T \nu^r,$$

and we have the result :

THEOREM VI (K) :—*In any trajectory the component of acceleration along the tangent to the trajectory is*

$$\ddot{s} \quad or \quad \dot{v} \quad or \quad T',$$

and the component along the principal normal is

$$\kappa \dot{s}^2 \quad or \quad \kappa v^2 \quad or \quad 2\kappa T,$$

where v is the velocity and κ the curvature of the trajectory.

If f_r denote the covariant components of the acceleration vector, we have†

(3.15) $$f_r = a_{rm} \ddot{q}^m + \begin{Bmatrix} mn \\ r \end{Bmatrix} \dot{q}^m \dot{q}^n = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^r}\right) - \frac{\partial T}{\partial q^r}.$$

To obtain an expression for the rate of change of kinetic energy, we differentiate

(3.16) $$v^2 = 2T = a_{mn} v^m v_n$$

with respect to t and obtain

(3.161) $$v\dot{v} = \dot{T} = a_{mn} \hat{v}^m v^n = a_{mn} f^m \dot{q}^n = f_n \dot{q}^n.$$

† *Cf.* WHITTAKER, *Analytical Dynamics* (1917), 39.

§ 3.2. *Law of unconstrained motion.*

The Lagrangian equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^r}\right) - \frac{\partial T}{\partial q^r} = Q_r, \tag{3.21}$$

where Q_r is the generalised force vector—covariant since $Q_r \delta q^r = \delta W$ is invariant. By (3.15) these equations can be written

$$f_r = Q_r, \tag{3.211}$$

or, in contravariant form,

$$f^r = Q^r, \tag{3.212}$$

where $Q^r = a^{rm} Q_m$ is the contravariant force vector. Thus we have the generalisation of the fundamental law of particle dynamics :

Theorem VII (K) :—*In a natural trajectory the acceleration vector is identical with the contravariant force vector.*

Let us now suppose that a certain line of force is a geodesic and let us think of a motion along this line of force defined by the equation

$$\ddot{s} = Q. \tag{3.22}$$

We wish to know whether this is a natural motion or not. Since the trajectory is a geodesic it has no curvature, and thus by (3.14) the acceleration vector is tangent to the trajectory and therefore co-directional with the force vector. By (3.22) the acceleration has the same magnitude as the force vector. Thus, by Theorem VII, the motion considered is a natural motion.

Conversely, if there exists a natural motion along a line of force, the acceleration vector must be tangent to the line of force. Thus the curvature of the line of force must be zero and it must therefore be a geodesic. Hence we have the result :

Theorem VIII (K) :—*Natural motion can take place along a line of force if and only if the line of force is a geodesic.*

For any natural motion we note that by (3.161) and (3.211)

$$\dot{T} = Q_n \dot{q}^n, \tag{3.23}$$

which merely states that the rate of increase of kinetic energy is equal to the rate of working of the forces acting on the system. To extract the full geometrical significance, we note that

$$Q_n \dot{q}^n = vQ \cos \phi, \tag{3.231}$$

where ϕ is the angle between the velocity vector and the contravariant force vector. Thus we may state the following theorem :

THEOREM IX (K) :—*In a natural motion the rate of increase of kinetic energy is equal to the product of the magnitude of the velocity vector, the magnitude of the force vector and the cosine of the angle between these vectors.*

We shall speak of any motion for which (3.23) is true as "satisfying the law of energy."

Equation (3.23) may also be written in the form

(3.24) $$T' = Q_n q^{n'} = Q \cos \phi.$$

Now if the force vector be resolved into components along N mutually perpendicular directions, of which one is tangent to the trajectory, $Q \cos \phi$ is equal to the component in the direction of the tangent. Thus, by Theorem VI, we have the result :

THEOREM X (K) :—*For any trajectory the tangential component of acceleration is equal to the tangential component of force if and only if the law of energy is satisfied.*

Equation (3.24) shows that, given a field of force and any curve in the manifold of configurations, there is an ∞^1 family of motions along the curve for which the law of energy is satisfied, the velocity and kinetic energy for such motions being defined as functions of the arc by the equations

(3.241) $$\tfrac{1}{2}v^2 \equiv T = T_0 + \int_{s_0}^{s} Q_n dq^n,$$

where T_0 is the value of T at the point $s = s_0$.

§ 3.3. BONNET'S *theorem.*

We now proceed to generalise the well-known theorem of OSSIAN BONNET† with regard to orbits under superimposed fields of force.

Let $Q^r_{(1)}, Q^r_{(2)}, \ldots, Q^r_{(M)}$ be M different force systems which can act either separately or all together on a given holonomic dynamical system. Let us suppose that there exists a curve C in the manifold of configurations which can be described under each of the several force systems acting alone. Let $T_{(1)}, T_{(2)}, \ldots, T_{(M)}$ be the kinetic energies in these several motions, each of these quantities being of course a function of the arc s of C measured from some fixed point. Then by (3.14) and (3.212)

(3.31) $$\begin{cases} Q^r_{(1)} = T'_{(1)}\lambda^r + 2\kappa T_{(1)}\nu^r, \\ Q^r_{(2)} = T'_{(2)}\lambda^r + 2\kappa T_{(2)}\nu^r, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Q^r_{(M)} = T'_{(M)}\lambda^r + 2\kappa T_{(M)}\nu^r. \end{cases}$$

Now let all the force systems act at once and let Q^r be the additional force system necessary to make the system describe the curve C with kinetic energy T given by

(3.32) $$T = T_{(1)} + T_{(2)} + \ldots + T_{(M)}.$$

† *Journal de Math.*, 9 (1844), 113 ; *cf.* WHITTAKER, *loc. cit.*, 94.

We have then

$$(3.33) \qquad Q^r + Q^r_{(1)} + Q^r_{(2)} + \ldots + Q^r_{(M)} = [T'_{(1)} + T'_{(2)} + \ldots + T'_{(M)}]\,\lambda^r + 2\kappa\,[T_{(1)} + T_{(2)} + \ldots + T_{(M)}]\,\nu^r,$$

so that, by (3.31), all the components of Q^r are zero, and we have the following result:

THEOREM XI (BONNET'S Theorem) :—*If a holonomic dynamical system can pass through a certain sequence of configurations under the separate influences of a number of force systems, then, if all the force systems act together, the system can pass through the same sequence of configurations with a kinetic energy equal to the sum of the kinetic energies which it had in the separate motions under the several force systems.*

§ 3.4. *Curvature of a trajectory.*

The curvature of a trajectory, defined as the curvature of its curve, is a purely geometrical or "statical" conception. For our purposes, however, it is more useful to employ the time t as an independent variable. We proceed to find an expression for the curvature of any trajectory, natural or unnatural.

Equation (3.123) may be written

$$(3.42) \qquad \dot{s}^2\kappa^r = f^r - \ddot{s}q^{r'},$$

and hence, multiplying by $\dot{s}^2$, we have

$$(3.421) \qquad 4T^2\kappa^r = 2Tf^r - \dot{T}\dot{q}^r.$$

Thus

$$(3.422) \qquad 16T^4\kappa^2 = a_{mn}(2Tf^m - \dot{T}\dot{q}^m)(2Tf^n - \dot{T}\dot{q}^n) = 4T^2f^2 - 4T\dot{T}a_{mn}f^m\dot{q}^n + 2T\dot{T}^2,$$

and hence, using (3.161), we obtain†

$$(3.43) \qquad \kappa^2 = \frac{f^2}{4T^2} - \frac{\dot{T}^2}{8T^3} = \frac{f^2 - \dot{v}^2}{v^4}.$$

If ϕ denotes the angle between the acceleration and velocity vectors, then

$$(3.44) \qquad \cos\phi = \frac{a^{mn}f^m\dot{q}^n}{f(2T)^{1/2}} = \frac{\dot{T}}{f(2T)^{1/2}},$$

or

$$(3.441) \qquad \dot{T}^2 = 2f^2T\cos^2\phi,$$

so that, substituting in (3.43) and taking the square root, we have

$$(3.45) \qquad \kappa = \frac{f\sin\phi}{2T}.$$

† An equivalent form was obtained by LIPKA, *Journal of Math. and Phys., Massachusetts Inst. of Technology*, 1 (1921), 33, for the curvature of a natural trajectory.

This result is really intuitively obvious from Theorem VI, since $f \sin \phi$ is the component of the acceleration vector in the direction of the principal normal.

We verify at once the well-known fact that a natural trajectory under no forces is a geodesic.† For, if there is no force, the acceleration is zero and therefore by (3.45) the curvature is zero, which is a sufficient condition for a geodesic in the case of a positive definite line-element.

§ 3.5. *Curvature relative to a natural trajectory.*

We now proceed to find an expression for the curvature of any trajectory C relative to the tangential natural trajectory C^*, having the same velocity vector at the point of contact (P).

From (3.421) we have

$$4T^2(\kappa^r - \kappa^{*r}) = 2Tf^r - \dot{T}\dot{q}^r - 2T^*f^{*r} + \dot{T}^*\dot{q}^{*r}, \tag{3.51}$$

where the unasterisked quantities refer to C, the asterisked to C^*. On future occasions we shall not trouble to explain this obvious notation. From the condition of identical velocities we have at P

$$\dot{q}^r = \dot{q}^{*r}, \qquad T = T^*. \tag{3.511}$$

Thus (3.51) may be written

$$4T^2(\kappa^r - \kappa^{*r}) = 2T(f^r - f^{*r}) - (\dot{T} - \dot{T}^*)\dot{q}^r. \tag{3.512}$$

Substituting in (2.34) we obtain

$$\begin{aligned} 16T^4[\kappa(C, C^*)]^2 = {} & 4T^2 a_{mn}(f^m - f^{*m})(f^n - f^{*n}) \\ & - 4T(\dot{T} - \dot{T}^*)a_{mn}(f^m - f^{*m})\dot{q}^n + (\dot{T} - \dot{T}^*)^2 a_{mn}\dot{q}^m\dot{q}^n. \end{aligned} \tag{3.52}$$

Now making use of the kinematical equation (3.161) and the corresponding asterisked equation, together with the dynamical equation of C^*, namely

$$f^{*r} = Q^r, \tag{3.53}$$

we obtain

$$8T^3[\kappa(C, C^*)]^2 = 2Ta_{mn}(f^m - Q^m)(f^n - Q^n) - (\dot{T} - a_{mn}Q^m\dot{q}^n)^2. \tag{3.54}$$

By introducing the force system, we have succeeded in getting rid of all explicit reference to the comparison trajectory C^*. Since $a_{mn}X^mX^n$ is a positive definite form for arbitrary values of X^r, the first term on the right-hand side is positive unless $f^r = Q^r$. The last bracket vanishes if the trajectory C satisfies the law of energy (3.23). Thus we have the result:

THEOREM XII (K) :—*If in any trajectory satisfying the law of energy the curvature at every instant relative to the natural trajectory having the same configuration and velocity vector is zero, then the trajectory is a natural trajectory.*

If we are dealing with a conservative system, in which case

$$Q_r = -\partial V/\partial q^r, \tag{3.55}$$

† *Cf.* RICCI and LEVI-CIVITA, *loc. cit.*, 179.

we have

(3.551) $$-a_{mn}Q^m\dot{q}^n = \dot{V}.$$

Thus, if H denotes the total energy, we have

(3.552) $$\dot{H} = \dot{T} + \dot{V} = \dot{T} - a_{mn}Q^m\dot{q}^n,$$

and (3.54) becomes

(3.56) $$8T^3[\kappa(C, C^*)]^2 = 2Ta_{mn}(f^m - Q^m)(f^n - Q^n) - \dot{H}^2.$$

§ 3.6. *The Principle of Least Curvature.*

Suppose that a holonomic dynamical system, whose kinetic energy is given by

(3.61) $$2T = a_{mn}\dot{q}^m\dot{q}^n,$$

is subjected to the external force system Q^r. Let M stationary constraints, holonomic or non-holonomic, be put on the system, expressed by the equations

(3.62) $$A_{(1)m}\,dq^m = A_{(2)m}\,dq^m = \ldots = A_{(M)m}\,dq^m = 0,$$

the coefficients being functions of position only. The equations of motion for the system so constrained are

(3.63) $$f^r = Q^r + P^r,$$

where P^r is the additional force vector introduced by the constraint, such that

(3.631) $$P_m\,\delta q^m = 0$$

for all displacements δq^r satisfying the equations of constraint (3.62).

We wish to consider three trajectories in the manifold of configurations, all having the same velocity vector at a point R (and therefore touching one another at that point):

(1) an arbitrary unnatural trajectory C, satisfying the conditions of constraint;
(2) the constrained natural trajectory C^*;
(3) the free natural trajectory C^{**}.

We have, by (3.54), at the point R

(3.64) $$8T^3[\kappa(C, C^{**})]^2 = 2Ta_{mn}(f^m - Q^m)(f^n - Q^n) - (\dot{T} - a_{mn}Q^m\dot{q}^n)^2,$$

(3.641) $$8T^3[\kappa(C^*, C^{**})]^2 = 2Ta_{mn}(f^{*m} - Q^m)(f^{*n} - Q^n),$$

use being made of the facts that the law of energy (3.23) is satisfied for C^* and that, at R, T^* is equal to T. Now the equations of motion for C^* are, as in (3.63),

(3.65) $$f^{*r} = Q^r + P^r,$$

and therefore, using (3.631),

(3.651) $$a_{mn}f^{*m}\,\delta q^n = a_{mn}(Q^m + P^m)\,\delta q^n = a_{mn}Q^m\,\delta q^n,$$

δq^r being any displacement satisfying the equations of constraint. But

$$\left\{ \begin{array}{l} A_{(1)m}\dot{q}^m = A_{(2)m}\dot{q}^m = \ldots = A_{(M)m}\dot{q}^m = 0, \\ A_{(1)m}\dot{q}^{*m} = A_{(2)m}\dot{q}^{*m} = \ldots = A_{(M)m}\dot{q}^{*m} = 0, \end{array} \right. \tag{3.66}$$

along C and C^* respectively, and therefore by differentiation with respect to t

$$\left\{ \begin{array}{l} A_{(1)m}f^m + A_{(1)mn}\dot{q}^m\dot{q}^n = \ldots = A_{(M)m}f^m + A_{(M)mn}\dot{q}^m\dot{q}^n = 0, \\ A_{(1)m}f^{*m} + A_{(1)mn}\dot{q}^{*m}\dot{q}^{*n} = \ldots = A_{(M)m}f^{*m} + A_{(M)mn}\dot{q}^{*m}\dot{q}^{*n} = 0, \end{array} \right. \tag{3.661}$$

with the usual notation for covariant derivatives. Hence, by subtraction, we have at R

$$A_{(1)m}(f^m - f^{*m}) = A_{(2)m}(f^m - f^{*m}) = \ldots = A_{(M)m}(f^m - f^{*m}) = 0. \tag{3.662}$$

Thus the displacement $\delta q^r = (f^r - f^{*r})\ \delta\tau$, where $\delta\tau$ is an arbitrary infinitesimal, satisfies the equations of constraint, and therefore, by (3.651),

$$a_{mn}f^{*m}(f^n - f^{*n}) = a_{mn}Q^m(f^n - f^{*n}). \tag{3.663}$$

Subtracting (3.641) from (3.64) we have

$$8T^3\{[\kappa(C, C^{**})]^2 - [\kappa(C^*, C^{**})]^2\} = 2Ta_{mn}[f^mf^n - f^{*m}f^{*n} - 2(f^m - f^{*m})Q^n] - (\dot{T} - a_{mn}Q^m\dot{q}^n)^2, \tag{3.67}$$

and substituting from (3.663) we find

$$8T^3\{[\kappa(C, C^{**})]^2 - [\kappa(C^*, C^{**})]^2\} = 2Ta_{mn}(f^m - f^{*m})(f^n - f^{*n}) - (\dot{T} - a^{mn}Q^m\dot{q}^n)^2. \tag{3.671}$$

Now the curve of the unnatural constrained trajectory C is perfectly arbitrary except in so far as it satisfied the conditions of constraint and has an assigned direction at R; the motion along C is restricted solely by an assigned velocity at R. Without further limiting the arbitrariness of the curve of C, let the motion along it be defined by (3.241), T_0 being the assigned kinetic energy at R; the law of energy (3.23) is then satisfied. The last term on the right-hand side of (3.671) vanishes and the remaining term is essentially positive. Thus

$$\kappa(C, C^{**}) > \kappa(C^*, C^{**}), \tag{3.672}$$

and we have the following theorem :

THEOREM XIII (K) (Least Curvature) :—*When a holonomic dynamical system is subjected to constraints, holonomic or non-holonomic, the natural constrained trajectory has, relative to the unconstrained natural trajectory with the same velocity vector, a smaller curvature than any other curve having the same tangent and satisfying the conditions of constraint.*

§ 3.7. *Law of motion in the case of one constraint, holonomic or non-holonomic.*

The preceding theorem embodies a descriptive law of motion for a system subjected to constraints. We shall now present the law of motion in a different and more explicit form, treating first the case of a single constraint, holonomic or non-holonomic.

Let Q^r be the external force vector and

(3.71) $$A_m \dot{q}^m = 0$$

the equation of constraint, the coefficients being functions of position. This equation implies that the velocity vector must always be perpendicular to the contravariant vector A^r, which we shall call the *constraint vector* and which is given at every point of the manifold. The equations of motion are (3.71) and

(3.72) $$f^r = Q^r + \theta A^r,$$

where θ is undetermined.† Differentiating (3.71) with respect to t we obtain

(3.73) $$A_m f^m + A_{mn} \dot{q}^m \dot{q}^n = 0,$$

where A_{mn} is the covariant derivative of A_m. Multiplying (3.72) by A_r, and summing as indicated, we have

(3.74) $$A_r f^r = A_r Q^r + \theta A_r A^r,$$

or, if A is the magnitude of the vector A^r,

(3.75) $$\theta A^2 = A_r f^r - A_r Q^r.$$

Hence, by (3.73),

(3.76) $$\theta A^2 = - A_{mn} \dot{q}^m \dot{q}^n - A_m Q^m.$$

Substituting for θ in (3.72) we obtain the equations of motion in the form

(3.77) $$f^r = Q^r - (A_m Q^m + A_{mn} \dot{q}^m \dot{q}^n) A^r / A^2.$$

But A^r/A is a unit vector; hence we have the following result:

THEOREM XIV (K) :—*When a holonomic dynamical system is subjected to a constraint, holonomic or non-holonomic, defined by the contravariant vector A^r, the system moves as if under the influence of an additional contravariant force vector co-directional with the constraint vector and of (directed) magnitude*

$$- (A_m Q^m + A_{mn} \dot{q}^m \dot{q}^n)/A.$$

§ 3.8. *Law of motion in the case of several constraints, holonomic or non-holonomic.*

Let us now suppose that instead of one constraint we have the following:

(3.81) $$A_{(1)m} \dot{q}^m = A_{(2)m} \dot{q}^m = \ldots = A_{(M)m} \dot{q}^m = 0.$$

Geometrically these equations imply that the velocity vector must be perpendicular

† *Cf.* WHITTAKER, *loc. cit.*, 215, where the covariant form is given.

to each of the M directions $A^r_{(1)}, A^r_{(2)}, \vdots\ , A^r_{(M)}$; thus the velocity vector at any point is constrained to lie in an element of $(N - M)$ dimensions. If these elements form in their totality $M\ \infty^1$ systems of surfaces, each of $(N - 1)$ dimensions, the equations (3.81) are integrable and the system is holonomic. This we do not suppose necessarily to be the case.

Let a system of M mutually perpendicular unit vectors $B^r_{(1)}, B^r_{(2)}, \ldots, B^r_{(M)}$ be selected in the elementary manifold of M dimensions defined by the vectors $A^r_{(1)}, A^r_{(2)}, \ldots A^r_{(M)}$. This is, of course a process that can be carried out in an infinity of ways, and the choice of method may be dictated by convenience in a particular problem. We may, however, generally proceed as follows. Put

$$B^r_{(1)} = A^r_{(1)}/A_{(1)}, \tag{3.811}$$

so that $B^r_{(1)}$ is a unit vector. Put

$$B^r_{(2)} = \xi_{(2,1)} B^r_{(1)} + \xi_{(2,2)} A^r_{(2)}, \tag{3.82}$$

where the ratio $\xi_{(2,1)} : \xi_{(2,2)}$ is defined by

$$\xi_{(2,1)} + \xi_{(2,2)}\, a_{mn} B^m_{(1)} A^n_{(2)} = 0, \tag{3.821}$$

and the magnitudes of $\xi_{(2,1)}$ and $\xi_{(2,2)}$ by

$$a_{mn} B^m_{(2)} B^n_{(2)} = 1. \tag{3.822}$$

Then $B^r_{(2)}$ is a unit vector perpendicular to $B^r_{(1)}$ and co-planar with $A^r_{(1)}$ and $A^r_{(2)}$. Put

$$B^r_{(3)} = \xi_{(3,1)} B^r_{(1)} + \xi_{(3,2)} B^r_{(2)} + \xi_{(3,3)} A^r_{(3)}, \tag{3.83}$$

where the ratios $\xi_{(3,1)} : \xi_{(3,2)} : \xi_{(3,3)}$ are defined by the equations

$$\begin{cases} \xi_{(3,1)} + \xi_{(3,3)}\, a_{mn} B^m_{(1)} A^n_{(3)} = 0, \\ \xi_{(3,2)} + \xi_{(3,3)}\, a_{mn} B^m_{(2)} A^n_{(3)} = 0, \end{cases} \tag{3.831}$$

and the magnitudes of these quantities by

$$a_{mn} B^m_{(3)} B^n_{(3)} = 1. \tag{3.832}$$

Then $B^r_{(3)}$ is a unit vector perpendicular to $B^r_{(1)}$ and $B^r_{(2)}$ and co-planar with $A^r_{(1)}$, $A^r_{(2)}$ and $A^r_{(3)}$. Proceeding in this manner we ultimately obtain M mutually perpendicular unit vectors $B^r_{(1)}, B^r_{(2)}, \ldots, B^r_{(M)}$ co-planar with $A^r_{(1)}, A^r_{(2)}, \ldots A^r_{(M)}$, so that

$$\begin{cases} B^r_{(1)} = \eta_{(1,1)} A^r_{(1)}, \\ B^r_{(2)} = \eta_{(2,1)} A^r_{(1)} + \eta_{(2,2)} A^r_{(2)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ B^r_{(M)} = \eta_{(M,1)} A^r_{(1)} + \eta_{(M,2)} A^r_{(2)} + \ldots + \eta_{(M,M)} A^r_{(M)}. \end{cases} \tag{3.84}$$

Now the velocity vector, being by (3.81) perpendicular to each of the A directions, must be perpendicular to every direction co-planar with these directions and therefore perpendicular to each of the B directions. Thus we may substitute for (3.81) the new equations of constraint

(3.85) $$B_{(1)m}\dot{q}^m = B_{(2)m}\dot{q}^m = \ldots = B_{(M)m}\dot{q}^m = 0.$$

The equations of motion are (3.85) together with

(3.86) $$f^r = Q^r + \theta^{(1)}B^r_{(1)} + \theta^{(2)}B^r_{(2)} + \ldots + \theta^{(M)}B^r_{(M)},$$

where $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(M)}$ are undetermined. Differentiation of (3.85) with respect to t gives

(3.87) $$B_{(1)m}f^m + B_{(1)mn}\dot{q}^m\dot{q}^n = \ldots = B_{(M)m}f^m + B_{(M)mn}\dot{q}^m\dot{q}^n = 0.$$

Multiplying (3.86) by $B_{(1)r}$ and summing as indicated, we obtain

(3.871) $$B_{(1)r}f^r = B_{(1)r}Q^r + \theta^{(1)},$$

using the fact that the B vectors are of unit magnitude and mutually perpendicular. Hence by (3.87)

(3.872) $$\theta^{(1)} = -B_{(1)m}Q^m - B_{(1)mn}\dot{q}^m\dot{q}^n.$$

Similar expressions may be obtained for $\theta^{(2)}, \ldots, \theta^{(M)}$. When we substitute in (3.86) we obtain the equations of motion

(3.88) $$\begin{aligned} f^r = Q^r &- (B_{(1)m}Q^m + B_{(1)mn}\dot{q}^m\dot{q}^n)\,B^r_{(1)} \\ &- (B_{(2)m}Q^m + B_{(2)mn}\dot{q}^m\dot{q}^n)\,B^r_{(2)} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &- (B_{(M)m}Q^m + B_{(M)mn}\dot{q}^m\dot{q}^n)\,B^r_{(M)}. \end{aligned}$$

Thus we have the result :

THEOREM XV (K) :—*When a holonomic system is subjected to M constraints, holonomic or non-holonomic, these constraints can be defined by M mutually perpendicular unit vectors $B^r_{(1)}, B^r_{(2)}, \ldots, B^r_{(M)}$. The system moves as if under the influence of an additional contravariant force vector co-planar with the B vectors and having in the directions of these vectors components*

$$-(B_{(1)m}Q^m + B_{(1)mn}\dot{q}^m\dot{q}^n),\ -(B_{(2)m}Q^m + B_{(2)mn}\dot{q}^m\dot{q}^n),\ \ldots,\ -(B_{(M)m}Q^m + B_{(M)mn}\dot{q}^m\dot{q}^n),$$

where $B_{(1)mn}, B_{(2)mn}, \ldots, B_{(M)mn}$ are the covariant derivatives of $B_{(1)m}, B_{(2)m}, \ldots B_{(M)m}$ respectively.

Equations (3.88) may also be written in the equivalent covariant form

(3.881) $$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^r}\right) - \frac{\partial T}{\partial q^r} = Q_r &- (B_{(1)m}Q^m + B_{(1)mn}\dot{q}^m\dot{q}^n)\,B_{(1)r} \\ &- (B_{(2)m}Q^m + B_{(2)mn}\dot{q}^m\dot{q}^n)\,B_{(2)r} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &- (B_{(M)m}Q^m + B_{(M)mn}\dot{q}^m\dot{q}^n)\,B_{(M)r}. \end{aligned}$$

We have here an extended and completely determinate form of LAGRANGE'S equations applicable to non-holonomic systems.

CHAPTER IV.

Study of the Manifold of Configurations with the Action Line-element.

$$ds^2 = 2(h - V) T dt^2 = (h - V) a_{mn} dq^m dq^n = g_{mn} dq^m dq^n.$$

§ 4.1. *Preliminary.*

In the previous chapter we had under consideration the kinematical line-element and all Christoffel symbols, curvatures, etc., were calculated for that line-element. Now we introduce the action line-element and, since it is necessary to use the previous theory to establish our results, we must be careful to avoid all confusion. We shall adhere to the general rule laid down in § 1.5 that expressions are ordinarily to be calculated for the line-element under consideration which for the present chapter is that of action. Wherever expressions calculated for the kinematical line-element have to be introduced, they will be marked with a subscript (K), *e.g.*,

$$(4.11) \qquad 2\begin{bmatrix} mn \\ r \end{bmatrix}_{(K)} = \frac{\partial a_{mr}}{\partial q^n} + \frac{\partial a_{nr}}{\partial q^m} - \frac{\partial a_{mn}}{\partial q^r}.$$

The general theory connected with the action line-element is of somewhat restricted interest for three reasons :

(1) The force system must be conservative.

(2) We cannot compare the results of different force systems without changing the line-element.

(3) It is awkward to compare two motions with different total energies.

The compensation for these restrictions lies in a certain greater simplicity in our results.

We shall only take into consideration motions with a total energy equal to the constant h occurring in the expression for the line-element. Thus to a curve there corresponds only one trajectory, time being defined in terms of arc-length (to within an additive constant) by the equation

$$(4.12) \qquad \dot{s}^2 = 2(h - V)^2.$$

Our discussions are not intended to take into consideration the singularities of this relation which arise when $V = h$, that is, when the kinetic energy vanishes.

Our theory is, in a sense, a statical theory and the expression of the laws should not involve the time.

§ 4.2. *Curvature vector of any curve ; law of unconstrained motion.*

For the covariant components of the curvature vector of any curve we have

$$(4.21) \qquad \kappa_r = g_{rm} q^{m''} + \begin{bmatrix} mn \\ r \end{bmatrix} q^{m'} q^{n'},$$

the accent now, of course, indicating differentiation with respect to the action line-element. But by simple calculation we find

$$(4.211)\qquad \begin{bmatrix} mn \\ r \end{bmatrix} = (h - V) \begin{bmatrix} mn \\ r \end{bmatrix}_{(K)} - \tfrac{1}{2}(a_{mr}V_n + a_{nr}V_m - a_{mn}V_r),$$

where $V_r = \partial V/\partial q^r$, and therefore (4.21) may be written

$$(4.22)\qquad \kappa_r = (h - V)\left(a_{rm}q^{m''} + \begin{bmatrix} mn \\ r \end{bmatrix}_{(K)} q^{m'}q^{n'}\right) - a_{mr}V_nq^{m'}q^{n'} + \tfrac{1}{2}V_ra_{mn}q^{m'}q^{n'}.$$

If we now introduce t as the independent parameter and remember that

$$(4.221)\qquad (h - V)\,a_{mn}q^{m'}q^{n'} = 1,$$

we obtain

$$(4.23)\qquad \kappa_r = \frac{h - V}{\dot{s}^2} f_r - \frac{h - V}{\dot{s}^3}\ddot{s}a_{rm}\dot{q}^m - \frac{a_{mr}V_n\dot{q}^m\dot{q}^n}{\dot{s}^2} + \tfrac{1}{2}\frac{V_r}{h - V},$$

f_r being the covariant acceleration vector as considered in Chapter III, and thus, since by (4.12)

$$(4.24)\qquad \dot{s}\,\ddot{s} = -2(h - V)\dot{V} = -2(h - V)V_n\dot{q}^n,$$

we have

$$(4.25)\qquad \kappa_r = \frac{h - V}{\dot{s}^2} f_r + \frac{2(h - V)^2}{\dot{s}^4} V_n\dot{q}^n a_{rm}\dot{q}^m - \frac{a_{mr}V_n\dot{q}^m\dot{q}^n}{\dot{s}^2} + \tfrac{1}{2}\frac{V_r}{h - V}.$$

Using (4.12) we see that the second and third terms on the right-hand side cancel and we have for the curvature vector κ_r of any trajectory

$$(4.26)\qquad 2(h - V)\,\kappa_r = f_r + V_r.$$

Thus, if the motion is natural and unconstrained, so that

$$(4.27)\qquad f_r = -V_r,$$

the components of the curvature vector vanish and the curve is a geodesic—a result which is well known.

§ 4.3. *Law of motion in the case of one constraint, holonomic or non-holonomic.*

When holonomic constraints are put on a system, the law of motion is simply that a natural curve is a geodesic in a submanifold defined by the constraints. If, however, the constraints are non-holonomic, no submanifold is defined and we have to look further for a geometrical statement for the law of motion.

Let us consider the case of one constraint given by the equation

$$(4.31)\qquad A_mq^{m'} = 0.$$

The equations of motion are, in the covariant form of (3.72),

(4.32) $$f_r = -V_r + \theta A_r.$$

Thus from (4.26) we have

(4.321) $$2(h-V)\kappa_r = \theta A_r,$$

or in contravariant form

(4.33) $$2(h-V)\kappa^r = \theta A^r.$$

We note that the principal normal of a natural constrained curve is co-directional with the constraint vector A^r.

Let us differentiate (4.31) with respect to s, obtaining

(4.34) $$A_m \kappa^m + A_{mn} q^{m'} q^{n'} = 0.$$

If we multiply (4.33) by A_r and sum as indicated, we obtain

(4.341) $$2(h-V) A_r \kappa^r = \theta A^2,$$

where A is the magnitude of the constraint vector. Thus, by (4.34),

(4.342) $$\theta A^2 = -2(h-V) A_{mn} q^{m'} q^{n'},$$

and if we substitute for θ in (4.33), we have the equations of the natural constrained curve in the form

(4.35) $$\kappa^r = -A_{mn} q^{m'} q^{n'} . A^r / A^2.$$

Hence

(4.351) $$\kappa^2 = (A_{mn} q^{m'} q^{n'} / A)^2,$$

and we have the result :

THEOREM XVI (A) :—*When a holonomic conservative system is subjected to a single constraint, holonomic or non-holonomic, the principal normal of the natural curve is co-directional with the constraint vector A^r and the curvature is*

$$\pm A_{mn} q^{m'} q^{n'} / A,$$

where A_{mn} is the covariant derivative of A_m.

§ 4.4. *Law of motion in the case of several constraints, holonomic or non-holonomic.*

The suggestion of the method immediately preceding, coupled with that of § 3.8, is so easy to follow that we shall not delay over the question of the law of motion in the case of several constraints. We shall merely state the result :

THEOREM XVII (A) :—*When a holonomic conservative system is subjected to M constraints, holonomic or non-holonomic, the complete constraint can be defined by M mutually perpendicular unit vectors $B^r_{(1)}, B^r_{(2)}, \ldots, B^r_{(M)}$. The natural curve is such that its* *principal normal is co-planar with these unit vectors and the contravariant curvature vector* κ^r *has components in their directions equal to*

$$-B_{(1)mn}q^{m'}q^{n'},\quad -B_{(2)mn}q^{m'}q^{n'},\quad \ldots,\quad -B_{(M)mn}q^{m'}q^{n'}.$$

The curvature of the natural curve is the square root of

$$(B_{(1)mn}q^{m'}q^{n'})^2 + (B_{(2)mn}q^{m'}q^{n'})^2 + \ldots + (B_{(M)mn}q^{m'}q^{n'})^2.$$

§ 4.5. *The Principle of Least Curvature.*

A principle of least curvature in the case of the action line-element follows easily from the similar principle for the kinematical line-element as established in § 3.6. Remembering that plain symbols refer to the action line-element and those marked (K) to the kinematical line-element, we have

$$ds = \sqrt{h - V}\,(ds)_{(K)}, \tag{4.51}$$

and therefore, when we examine the definition of relative curvature as given in § 2.3, we see at once that

$$\kappa\,(C, C^*) = \frac{1}{\sqrt{h - V}}\,[\kappa\,(C, C^*)]_{(K)}, \tag{4.52}$$

where C and C^* are any two curves touching one another.

Now let us suppose that a holonomic conservative system is subjected to certain constraints, holonomic or non-holonomic. Let us think of three curves in the manifold of configurations, touching one another at a point P:

(1) any unnatural curve C, satisfying the conditions of constraint;
(2) the natural constrained curve C^*;
(3) the natural unconstrained curve C^{**}.

Remembering that we are considering only motions with a total energy h, so that the velocities of the natural motions on C^* and C^{**} are equal at P, we can apply Theorem XIII (§ 3.6) to obtain the inequality for curvatures with respect to the kinematical line-element

$$[\kappa\,(C^*, C^{**})]_{(K)} < [\kappa\,(C, C^{**})]_{(K)}. \tag{4.53}$$

Hence by (4.52)

$$\kappa\,(C^*, C^{**}) < \kappa\,(C, C^{**}). \tag{4.531}$$

But C^{**} is a geodesic with respect to the action line-element and curvature relative to it is curvature in the absolute sense. Thus we have the result:

THEOREM XVIII (A) (Least Curvature):—*When a holonomic conservative system is subjected to constraints, holonomic or non-holonomic, the natural constrained curve has a smaller curvature than any other curve having the same tangent and satisfying the conditions of constraint.*

CHAPTER V.

Ignorability of Co-ordinates with the Kinematical Line-element.

§ 5.1. *Statement of the problem.*

A conservative dynamical system is said to have an *ignorable* co-ordinate when that co-ordinate does not occur explicitly in the kinetic potential L. The importance of ignorable co-ordinates lies in the fact that the number of degrees of freedom of a dynamical system can be reduced by a number equal to the number of ignorable co-ordinates.†

If

(5.11) $$L = T - V = \tfrac{1}{2} a_{mn} \dot{q}^m \dot{q}^n - V,$$

the necessary and sufficient conditions that the co-ordinates $q^1, q^2, \ldots, q^M$ should be ignorable are

(5.12) $$\frac{\partial a_{mn}}{\partial q^1} = \frac{\partial a_{mn}}{\partial q^2} = \ldots = \frac{\partial a_{mn}}{\partial q^M} = 0,$$

and

(5.13) $$\frac{\partial V}{\partial q^1} = \frac{\partial V}{\partial q^2} = \ldots = \frac{\partial V}{\partial q^M} = 0.$$

The existence of ignorable co-ordinates is clearly dependent on the particular system of co-ordinates q^r used to specify the configurations of the system. For example, in the case of a particle in a plane, attracted to a fixed centre by a force depending on the distance only, the argument θ is an ignorable co-ordinate when the configurations are specified by polar co-ordinates (r, θ), but when cartesian co-ordinates are employed there is no ignorable co-ordinate. Thus if we take a dynamical system with ignorable co-ordinates and transform to a new system of co-ordinates, it will in general happen that none of the new co-ordinates are ignorable.

An important problem presents itself: being given a dynamical system, to determine whether it is possible to find a co-ordinate system for which some of the co-ordinates are ignorable. An answer to this problem in terms of rigid-body displacements in the manifold of configurations has been given by Levy‡ and the existence of a linear integral has been discussed by Ricci and Levi-Civita,§ but these treatments do not seem to afford tests by means of which we can determine by mere calculation whether a system admits ignorable co-ordinates or not.

In the present paper we shall not attempt a solution of the general problem of determining the maximum number of ignorable co-ordinates which a system may admit. We shall confine our attention to the determination of necessary and sufficient conditions that a system with N degrees of freedom should admit $(N - 1)$ ignorable co-ordinates. The special case of two degrees of freedom is of such relative simplicity that we shall discuss it separately.

† *Cf.* Whittaker, *loc. cit.*, 54.

‡ *Comptes rendus*, 86 (1878), 463 and 875.

§ *Loc. cit.*, 179.

§ 5.2. *The case of two degrees of freedom.*

Let q^1 be ignorable, so that

(5.21) $$\frac{\partial a_{mn}}{\partial q^1} = 0$$

and

(5.22) $$\frac{\partial V}{\partial q^1} = 0.$$

This latter equation tells us that the parametric lines of q^1 are equipotential curves. Let C and C^* be neighbouring equipotential curves and let P and P^* be points on them corresponding to equal values of q^1. Let δq^2 denote the increment in q^2 in passing from P to P^*. Then

(5.23) $$(PP^*)^2 = a_{22}\,(\delta q^2)^2,$$

which is independent of q^1 by virtue of (5.21). Thus PP^* is of constant length. Also, if θ denotes the angle between PP^* and C

(5.231) $$\cos\theta = \frac{a_{12} q^{1'}\,\delta q^2}{PP^*} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}},$$

and this also is independent of q^1. Thus, if P^*N is the perpendicular let fall from P^* on C, P^*N is of constant length, being equal to $PP^*\sin\theta$. Thus *the equipotential curves are parallel.*

The components of the curvature vector of C are

(5.232) $$\kappa_r = a_{r1} q^{1''} + \begin{bmatrix} 11 \\ r \end{bmatrix} (q^{1'})^2,$$

where

(5.233) $$a_{11}\,(q^{1'})^2 = 1, \quad q^{1''} = 0.$$

Thus, using (5.21), we have

(5.234) $$\begin{cases} \kappa_1 = 0, \\ \kappa_2 = -\tfrac{1}{2}\,\dfrac{1}{a_{11}}\,\dfrac{\partial a_{11}}{\partial q^2}. \end{cases}$$

Hence the curvature of C is given by

(5.24) $$\pm\,\kappa = -\tfrac{1}{2}\,\frac{\sqrt{a^{22}}}{a_{11}}\,\frac{\partial a_{11}}{\partial q^2},$$

which is independent of q^1, and thus *the equipotential curves are each of constant curvature.*

We shall now show that if the equipotentials are parallel curves, each of constant curvature, a co-ordinate system exists such that one of the co-ordinates is ignorable.

Let us take a *G.O.T.* (q^2) co-ordinate system q^r (see § 2.5), the family of parallel equipotential curves having the equations $q^2 = constant$. Further, let the value of q^1 on the equipotential curve $q^2 = 0$ be equal to the arc measured from some fixed point. Then we have everywhere

(5.25) $$a_{12} = 0, \qquad a_{22} = 1,$$

and when q^2 is zero

(5.251) $$(a_{11})_{q^2=0} = 1.$$

In order to establish the ignorability of q^1 it is sufficient to prove

(5.26) $$\frac{\partial a_{11}}{\partial q^1} = 0.$$

Now the curvature of an equipotential curve $q^2 = constant$ is, as a very particular case of (2.57),

(5.27) $$\kappa = -\tfrac{1}{2}\frac{\partial a_{11}}{\partial q^2}(q^{1'})^2 = -\tfrac{1}{2}\frac{1}{a_{11}}\frac{\partial a_{11}}{\partial q^2}.$$

But we are given that κ is independent of q^1; therefore

(5.271) $$\frac{\partial^2}{\partial q^1 \partial q^2}(\log a_{11}) = 0.$$

But, since $\partial a_{11}/\partial q^1$ vanishes for $q^2 = 0$, we have

(5.272) $$\left[\frac{\partial}{\partial q^1}(\log a_{11})\right]_{q^2=0} = 0,$$

and therefore, by (5.271),

(5.273) $$\frac{\partial}{\partial q^1}(\log a_{11}) = 0$$

in general. Thus (5.26) is proved.

Remembering that the lines of force are the orthogonal trajectories of the equipotentials and that they are geodesics if and only if the equipotentials are parallel, we have the following result :

THEOREM XIX (K) :—*In order that it may be possible to find for a conservative dynamical system with two degrees of freedom a system of co-ordinates for which one co-ordinate is ignorable, it is necessary and sufficient that the equipotentials should be parallel curves each of constant curvature, or, equivalently, that the lines of force should be geodesics and the equipotential curves each of constant curvature. The parametric lines of the ignorable co-ordinate are the equipotential curves.*

We deduce an interesting result in the case where the kinetic energy is homaloidal, *i.e.*, reducible to a sum of squares of velocities, as in the case of a particle in the Euclidean plane. In this case it is possible to find an ignorable co-ordinate if and only if the equipotential curves form a system of concentric circles.

We shall now proceed to establish analytical conditions by means of which it may be ascertained by mere calculation whether or not a system with two degrees of freedom admits an ignorable co-ordinate. In what follows q^r is a perfectly general co-ordinate system.

From Theorem IV (§ 2.6) we know that in order that the orthogonal trajectories of the equipotential curves may be geodesics it is necessary and sufficient that the single condition

(5.28) $$\frac{a^{mn} V_{1m} V_n}{V_1} = \frac{a^{mn} V_{2m} V_n}{V_2}$$

should be satisfied. We have also from Theorem III the equivalent conditions

(5.281) $$\lambda_{rs}\lambda^s = 0,$$

where λ^r is the unit vector everywhere perpendicular to the equipotentials.

Assuming that the above conditions are satisfied, so that the equipotential curves are parallel, let us seek an analytical expression for the condition that the curvature of each equipotential curve should be constant. For this purpose we shall use the formula of Theorem I. (§ 2.5), which gives the curvature of any equipotential in the form

(5.282) $$\kappa = -\lambda_{rs} q^{r'} q^{s'}.$$

Differentiating this equation with respect to the arc of the equipotential, we obtain

(5.283) $$\kappa' = -\lambda_{rst} q^{r'} q^{s'} q^{t'} - \lambda_{rs} \kappa^r q^{s'} - \lambda_{rs} q^{r'} \kappa^s.$$

Substituting for κ^r from (2.571) and using (5.281) and the identical relation (2.662), we see that the last two terms vanish and we have

(5.284) $$\kappa' = -\lambda_{rst} q^{r'} q^{s'} q^{t'}.$$

But along the equipotential

(5.285) $$\lambda_m q^{m'} = 0,$$

and therefore, since there are only two dimensions,

(5.286) $$q^{1'} = \theta\lambda_2, \qquad q^{2'} = -\theta\lambda_1,$$

where θ is undetermined. On substitution of these values in (5.284) we get a value for κ' expressed as a function of position and we have the result :

THEOREM XX (K) :—*In order that it may be possible to find for a conservative dynamical system with two degrees of freedom a system of co-ordinates for which one co-ordinate is ignorable, it is necessary and sufficient that*

(5.287) $$\frac{a^{mn} V_{1m} V_n}{V_1} = \frac{a^{mn} V_{2m} V_n}{V_2},$$

and that

(5.288) $$\lambda_{111} (V_2)^3 - (\lambda_{112} + \lambda_{121} + \lambda_{211}) (V_2)^2 V_1 + (\lambda_{221} + \lambda_{212} + \lambda_{122}) V_2 (V_1)^2 - \lambda_{222} (V_1)^3 = 0,$$

where $\lambda_r = V_r/(a^{mn} V_m V_n)^{1/2}$, $V_r = \partial V/\partial q^r$ *and* λ_{rs}, λ_{rst} *are covariant derivatives.*

§ 5.3. *Necessary geometrical conditions for the admissibility of* $(N - 1)$ *ignorable co-ordinates.*

Passing on now to the general case of N degrees of freedom, we shall find necessary conditions in order that it may be possible to choose co-ordinates such that all but one are ignorable.

Suppose that a co-ordinate system $q^1, q^2, \ldots, q^N$ exists such that all the co-ordinates are ignorable except q^N. Then

(5.31) $$\frac{\partial a_{mn}}{\partial q^\rho} = 0,$$

and

(5.32) $$\frac{\partial V}{\partial q^\rho} = 0,$$

where the convention with respect to Greek indices (see § 2.1) is to be remembered.

Referred to this co-ordinate system, the equipotential surfaces have the equations $q^N =$ *constant.* Let P $(q^1, q^2, \ldots, q^N)$ be a point situated on the equipotential surface S and let P^* $(q^1, q^2, \ldots, q^N + \delta q^N)$ be the point where a neighbouring equipotential surface S^* is met by the parametric line of q^N passing through P. Then

(5.33) $$(PP^*)^2 = a_{NN}\,(\delta q^N)^2,$$

which is independent of $q^1, q^2, \ldots, q^{N-1}$. Thus two adjacent equipotential surfaces make equal intercepts on all parametric lines of q^N. Now the unit vector normal to S has covariant components $(0, 0, \ldots, 1/\sqrt{a^{NN}})$ and hence the angle θ between PP^* and the normal to S is given by

(5.34) $$\cos\theta = \frac{1}{\sqrt{a_{NN}a^{NN}}},$$

which is independent of $q^1, q^2, \ldots, q^{N-1}$. Thus the normal distance between S and S^*, being equal to $PP^*\cos\theta$, is also independent of these co-ordinates and S and S^* are parallel surfaces; therefore *the lines of force are geodesics.*

Let us now find the curvature of a surface geodesic of an equipotential S. The equations of the geodesic are

(5.35) $$a_{\rho\mu}q^{\mu''} + \begin{bmatrix}\mu\nu\\ \rho\end{bmatrix} q^{\mu'}q^{\nu'} = 0.$$

But by (5.31) the Christoffel symbols occurring in this equation all vanish and (5.35) is equivalent to

(5.351) $$q^{\mu''} = 0.$$

For the calculation of the curvature with respect to the manifold of configurations we have

(5.352) $$\kappa_r = a_{rm}q^{m''} + \begin{bmatrix}mn\\ r\end{bmatrix} q^{m'}q^{n'} = \begin{bmatrix}\mu\nu\\ r\end{bmatrix} q^{\mu'}q^{\nu'}.$$

Thus, by (5.31),

(5.353) $$\begin{cases} \kappa_\rho = 0, \\ \kappa_N = \begin{bmatrix} \mu\nu \\ N \end{bmatrix} q^{\mu'} q^{\nu'} = -\frac{1}{2}\frac{\partial a_{\mu\nu}}{\partial q^N} q^{\mu'} q^{\nu'}, \end{cases}$$

and we have

(5.354) $$\kappa = -\tfrac{1}{2}\sqrt{a^{NN}}\,\frac{\partial a_{\mu\nu}}{\partial q^N}\, q^{\mu'} q^{\nu'}.$$

Now let us pass from a geodesic C of the equipotential surface S through the point P to another geodesic D of the same surface S, passing through a neighbouring point Q in a direction derived from the direction of C at P by parallel propagation with respect to the metric of S (not of the manifold of configurations). We shall call such a geodesic D *neighbouring and parallel* to C. We may write $(q^1, \ldots, q^{N-1}, q^N)$ for the co-ordinates of P, $(q^1 + \delta q^1, \ldots, q^{N-1} + \delta q^{N-1}, q^N)$ for the co-ordinates of Q, $(q^{1'}, \ldots, q^{N-1'}, 0)$ for the unit contravariant vector at P having the direction of C and $(q^{1'} + \delta q^{1'}, \ldots, q^{N-1'} + \delta q^{N-1'}, 0)$ for the unit contravariant vector at Q having the direction of D, where

(5.355) $$a_{\rho\mu}\delta q^{\mu'} + \begin{bmatrix} \mu\nu \\ \rho \end{bmatrix} q^{\mu'}\delta q^{\nu} = 0.$$

By (5.31) these reduce to

(5.356) $$\delta q^{\rho'} = 0.$$

If $\delta\kappa$ denotes the difference between the curvature of D at Q and the curvature of C at P, we have from (5.354)

(5.357) $$\delta\kappa = -\tfrac{1}{2}\frac{\partial}{\partial q^\rho}\left[\sqrt{a^{NN}}\,\frac{\partial a_{\mu\nu}}{\partial q^N}\, q^{\mu'} q^{\nu'}\right]\delta q^\rho - \sqrt{a^{NN}}\,\frac{\partial a_{\mu\nu}}{\partial q^N}\, q^{\mu'}\delta q^{\nu'},$$

so that, by (5.31) and (5.356),

(5.358) $$\delta\kappa = 0.$$

Thus *the curvature of C at P is equal to the curvature of D at Q.* Since the tangential direction is propagated parallelly along the geodesic, it follows as a particular case that the curvature of every geodesic of an equipotential surface is constant along its length.

There is a third type of necessary condition, non-existent in the case of two degrees of freedom. The line-element of any equipotential surface S is

(5.36) $$ds^2 = a_{\mu\nu} dq^\mu dq^\nu,$$

where $a_{\mu\nu}$ are constants over S. Thus *S is a homaloidal manifold, i.e., ds^2* is transformable into a sum of squares of differentials of co-ordinates.

We may state the result :

THEOREM XXI (K).—*If for a conservative dynamical system a system of co-ordinates exists such that all the co-ordinates but one are ignorable, then*

(α) *the equipotential surfaces are parallel and the lines of force are geodesics ;*

(β) *the curvatures, relative to the manifold of configurations, of any pair of neighbouring and parallel geodesics of an equipotential surface are equal and the curvature of any geodesic of an equipotential surface is constant along its length ;*

(γ) *every equipotential surface is a homaloidal manifold.*

Applying these tests to simple types of equipotential surfaces in the case of the motion of a particle in three-dimensional Euclidean space, we may note that a family of concentric spheres satisfies (α) and (β) but not (γ), a family of coaxal equiangular right circular cones satisfies (α) and (γ) but not (β), while a family of coaxal circular cylinders satisfies (α), (β) and (γ).

§ 5.4. *Sufficient geometrical conditions for the admissibility of* ($N-1$) *ignorable co-ordinates.*

We shall now proceed to prove the following theorem :—

THEOREM XXII (K)—*If for a conservative dynamical system the following conditions are satisfied :—*

(α) *the lines of force are geodesics ;*

(β) *the curvatures, relative to the manifold of configurations, of every pair of neighbouring and parallel geodesics of an equipotential surface are equal, and this is true for every equipotential surface ;*

(γ) *there exists a homaloidal equipotential surface ;*

then a system of co-ordinates exists such that all the co-ordinates but one are ignorable.

Let us take a *G.O.T.* (q^N) co-ordinate system such that the homaloidal equipotential surface S_0 has the equation $q^N = 0$. Then, by (α), the equations of all the equipotential surfaces are $q^N = constant$. The choice of $q^1, q^2, \ldots, q^{N-1}$ is still arbitrary on one of the equipotential surfaces ; let us choose these co-ordinates such that on S_0 the fundamental tensor has constant components. Thus

(5.41) $$\left(\frac{\partial a_{\mu\nu}}{\partial q^\rho}\right)_{q^N=0} = 0.$$

Now since

(5.411) $$a_{N\rho} = 0, \quad a_{NN} = 1,$$

it is only necessary to prove that

(5.412) $$\frac{\partial a_{\mu\nu}}{\partial q^\rho} = 0$$

everywhere, in order to establish the ignorability of all the co-ordinates but q^N, the conditions

(5.413) $$\frac{\partial V}{\partial q^\rho} = 0$$

being already satisfied by the choice of co-ordinates.

The curvature of any geodesic of an equipotential surface is, by (2.57),

$$\kappa = -\tfrac{1}{2}\frac{\partial a_{\mu\nu}}{\partial q^N} q^{\mu'} q^{\nu'}, \tag{5.42}$$

and, if $\delta\kappa$ denotes the increment in κ in passing to a neighbouring and parallel geodesic, we have

$$\delta\kappa = -\tfrac{1}{2}\frac{\partial^2 a_{\mu\nu}}{\partial q^\tau \partial q^N} q^{\mu'} q^{\nu'} \delta q^\tau - \frac{\partial a_{\mu\nu}}{\partial q^N} q^{\mu'} \delta q^{\nu'}, \tag{5.421}$$

where†

$$\delta q^{\rho'} + \left\{ \begin{matrix} \sigma\tau \\ \rho \end{matrix} \right\} q^{\sigma'} \delta q^\tau = 0. \tag{5.422}$$

Thus

$$\delta\kappa = -\left(\tfrac{1}{2}\frac{\partial^2 a_{\mu\nu}}{\partial q^\tau \partial q^N} - \frac{\partial a_{\mu\rho}}{\partial q^N} \left\{ \begin{matrix} \nu\tau \\ \rho \end{matrix} \right\} \right) q^{\mu'} q^{\nu'} \delta q^\tau. \tag{5.423}$$

But, by the hypothesis (β), $\delta\kappa$ is zero for arbitrary values of the infinitesimal displacement δq^ρ and of the direction $q^{\rho'}$; therefore

$$\frac{\partial^2 a_{\mu\nu}}{\partial q^\tau \partial q^N} = \frac{\partial a_{\mu\rho}}{\partial q^N} \left\{ \begin{matrix} \nu\tau \\ \rho \end{matrix} \right\} + \frac{\partial a_{\nu\rho}}{\partial q^N} \left\{ \begin{matrix} \mu\tau \\ \rho \end{matrix} \right\}, \tag{5.424}$$

or, more explicitly,

$$\frac{\partial}{\partial q^N}\frac{\partial a_{\mu\nu}}{\partial q^\tau} = \tfrac{1}{2}\left(\frac{\partial a_{\nu\sigma}}{\partial q^\tau} + \frac{\partial a_{\tau\sigma}}{\partial q^\nu} - \frac{\partial a_{\nu\tau}}{\partial q^\sigma} \right) a^{\rho\sigma} \frac{\partial a_{\mu\rho}}{\partial q^N} + \tfrac{1}{2}\left(\frac{\partial a_{\mu\sigma}}{\partial q^\tau} + \frac{\partial a_{\tau\sigma}}{\partial q^\mu} - \frac{\partial a_{\mu\tau}}{\partial q^\sigma} \right) a^{\rho\sigma} \frac{\partial a_{\mu\rho}}{\partial q^N}. \tag{5.425}$$

Now, thinking of the determination of the quantities $\partial a_{\mu\nu}/\partial q^\tau$ as functions of q^N, we have here a system of linear differential equations of the first order; thus, by virtue of (5.41), it follows that (5.412) is true and the theorem is proved.

§ 5.5. *Necessary and sufficient analytical conditions for the admissibility of* ($N-1$) *ignorable co-ordinates.*

We have in Theorems XXI and XXII geometrical statements of necessary and sufficient conditions for the existence of a set of co-ordinates of which all but one are ignorable. We shall now supply the equivalent analytical conditions.

We have already in Theorem IV (§ 2.6) expressed necessary and sufficient analytical conditions that the lines of force should be geodesics. We shall therefore pass on to the determination of necessary and sufficient analytical conditions for the equality of curvatures of neighbouring and parallel geodesics of an equipotential surface, assuming that the equipotential surfaces are parallel. In what follows q^r is a perfectly general co-ordinate system.

† The Christoffel symbols occurring in this equation are to be calculated for the fundamental tensor of the surface S. However, the co-ordinate system being *G.O.T.* (q^N), it is easily seen that they have the same values whether calculated for the fundamental tensor of S or for the fundamental tensor of the manifold of configurations.

By Theorem I (§ 2.5) the curvature of any geodesic of one of the equipotential surfaces $V(q^1, q^2, \ldots, q^N) = constant$ is

(5.51) $$\kappa = -\lambda_{rs} q^{r'} q^{s'},$$

where $\lambda_r = V_r/(a^{mn} V_m V_n)^{\frac{1}{2}}$. If p^r is a *G.O.T.* (p^N) co-ordinate system for which the equipotential surfaces have the equations $p^N = constant$, we have

(5.511) $$\kappa = -\mu_{rs} p^{r'} p^{s'} = -\mu_{\rho\sigma} p^{\rho'} p^{\sigma'},$$

where μ_r are the components of λ_r when referred to the p^r co-ordinate system, so that

(5.512) $$\mu_\rho = 0, \quad \mu_N = 1.$$

Then, passing to a neighbouring and parallel geodesic of the same equipotential surface, we find

(5.513) $$\delta\kappa = -\left(\frac{\partial\mu_{\rho\sigma}}{\partial p^\tau} - \begin{Bmatrix}\sigma\tau \\ \nu\end{Bmatrix}_{(p)} \mu_{\rho\nu} - \begin{Bmatrix}\rho\tau \\ \nu\end{Bmatrix}_{(p)} \mu_{\nu\sigma}\right) p^{\rho'} p^{\sigma'} \delta p^\tau,$$

where the Christoffel symbols for the p^r co-ordinate system are marked with a subscript (p). But we easily find that

(5.514) $$\mu_{\rho N} = \mu_{N\rho} = 0,$$

and therefore

(5.515) $$\delta\kappa = -\left(\frac{\partial\mu_{\rho\sigma}}{\partial p^\tau} - \begin{Bmatrix}\sigma\tau \\ n\end{Bmatrix}_{(p)} \mu_{\rho n} - \begin{Bmatrix}\rho\tau \\ n\end{Bmatrix}_{(p)} \mu_{n\sigma}\right) p^{\rho'} p^{\sigma'} \delta p^\tau,$$
$$= -\mu_{\rho\sigma\tau} p^{\rho'} p^{\sigma'} \delta p^\tau,$$

where $\mu_{\rho\sigma\tau}$ is the covariant derivative of $\mu_{\rho\sigma}$ for the co-ordinate system p^r. But $\mu_{\rho\sigma}$ is symmetric in ρ and σ (*cf.* 2.586), and hence it is easily seen that the necessary and sufficient conditions for the vanishing of $\delta\kappa$ are

(5.516) $$\mu_{\rho\sigma\tau} = 0.$$

But

(5.517) $$\mu_{\rho\sigma\tau} = \lambda_{rst} \frac{\partial q^r}{\partial p^\rho} \frac{\partial q^s}{\partial p^\sigma} \frac{\partial q^t}{\partial p^\tau},$$

and therefore, since $\partial q^r/\partial p^\rho$ is a general vector in the equipotential surface, we may state our result in the following form:

THEOREM XXIII (K)—*Being given a family of parallel surfaces whose equations are* $V(q^1, q^2, \ldots, q^N) = constant$, *in order that the curvatures of every pair of neighbouring and parallel goedesics of each of these surfaces may be equal, it is necessary and sufficient that*

(5.52) $$\lambda_{rst} \xi^r \eta^s \zeta^t = 0,$$

(*where* $\lambda_r = V_r/(a^{mn} V_m V_n)^{\frac{1}{2}}$ *and* λ_{rst} *is the second covariant derivative of* λ_r) *for all values of the vectors* ξ^r, η^r, ζ^r *consistent with*

(5.521) $$V_m \xi^m = V_m \eta^m = V_m \zeta^m = 0.$$

We pass on now to necessary and sufficient conditions that a surface of a family of parallel surfaces $V(q^1, q^2, \ldots, q^N) = constant$ should be homaloidal.

In order that a manifold of N dimensions should be homaloidal, it is necessary and sufficient that the curvature tensor should vanish—the curvature tensor being defined (as usual) as

(5.53) $$G_{mnst} = \frac{\partial}{\partial x^s}\begin{bmatrix} nt \\ m \end{bmatrix} - \frac{\partial}{\partial x^t}\begin{bmatrix} ns \\ m \end{bmatrix} - g^{uv}\left\{\begin{bmatrix} nt \\ u \end{bmatrix}\begin{bmatrix} ms \\ v \end{bmatrix} - \begin{bmatrix} ns \\ u \end{bmatrix}\begin{bmatrix} mt \\ v \end{bmatrix}\right\},$$

for a co-ordinate system x^r and a fundamental tensor g_{mn}.

Let us take a *G.O.T.* (p^N) co-ordinate system p^r for which the equations of the family of surfaces are $p^N = constant$, and let b_{mn} be the corresponding fundamental tensor. It is easily established† by direct calculation that

(5.531) $$(G_{\mu\nu\sigma\tau})_{S(p)} = (G_{\mu\nu\sigma\tau})_{(p)} - \tfrac{1}{4}\left(\frac{\partial b_{\mu\tau}}{\partial p^N}\frac{\partial b_{\nu\sigma}}{\partial p^N} - \frac{\partial b_{\mu\sigma}}{\partial p^N}\frac{\partial b_{\nu\tau}}{\partial p^N}\right),$$

where the subscript $S(p)$ identifies the curvature tensor of the equipotential surface S, calculated for the co-ordinate system p^ρ, and the subscript (p) identifies the curvature tensor of the manifold of configurations, calculated for the co-ordinate system p^r. Thus the necessary and sufficient conditions that S should be homaloidal are

(5.532) $$(G_{\mu\nu\sigma\tau})_{(p)} - \tfrac{1}{4}\left(\frac{\partial b_{\mu\tau}}{\partial p^N}\frac{\partial b_{\nu\sigma}}{\partial p^N} - \frac{\partial b_{\mu\sigma}}{\partial p^N}\frac{\partial b_{\nu\tau}}{\partial p^N}\right) = 0.$$

But, if we compare with (2.586), this may be written

(5.533) $$(G_{\mu\nu\sigma\tau})_{(p)} - (\mu_{\mu\tau}\mu_{\nu\sigma} - \mu_{\mu\sigma}\mu_{\nu\tau}) = 0$$

or

(5.534) $$[G_{mnst} - (\lambda_{mt}\lambda_{ns} - \lambda_{ms}\lambda_{nt})]\frac{\partial q^m}{\partial p^\mu}\frac{\partial q^n}{\partial p^\nu}\frac{\partial q^s}{\partial p^\sigma}\frac{\partial q^t}{\partial p^\tau} = 0.$$

Thus we have the result :

THEOREM XXIV (K).—*In order that a surface of a family of parallel surfaces whose equations are* $V(q^1, q^2, \ldots, q^N) = constant$ *should be homaloidal, it is necessary and sufficient that*

(5.54) $$[G_{mnst} - (\lambda_{mt}\lambda_{ns} - \lambda_{ms}\lambda_{nt})]\xi^m\eta^n\zeta^s\omega^t = 0,$$

at all points of the surface and for all values of the vectors ξ^r, η^r, ζ^r, ω^r *consistent with*

(5.541) $$V_m\xi^m = V_m\eta^m = V_m\zeta^m = V_m\omega^m = 0,$$

where G_{mnst} *is the curvature tensor of the manifold containing the family and*

$$\lambda_r = V_r/(a^{mn}V_mV_n)^{1/2}, \; V_r = \partial V/\partial q^r;$$

λ_{rs} *is the covariant derivative of* λ_r.

† BIANCHI, *loc. cit.*, 452.

Let us now group our established geometrical results into a single dynamical theorem :

THEOREM XXV (K)—*In order that it may be possible in the case of a conservative dynamical system to find a system of co-ordinates for which all co-ordinates but one are ignorable, it is necessary and sufficient that the following conditions should be satisfied :—*

$$(\alpha) \qquad \frac{a^{mn} V_{rm} V_n}{V_r} = \frac{a^{mn} V_{sm} V_n}{V_s},$$

at all points of the manifold of configurations ;

$$(\beta) \qquad \lambda_{rst} \xi^r \eta^s \zeta^t = 0,$$

at all points of the manifold of configurations and for all values of ξ^r, η^r, ζ^r *such that*

$$V_m \xi^m = V_m \eta^m = V_m \zeta^m = 0 ;$$

$$(\gamma) \qquad [G_{mnst} - (\lambda_{mt}\lambda_{ns} - \lambda_{ms}\lambda_{nt})]\, \xi^m \eta^n \zeta^s \omega^t = 0,$$

at all points of at least one equipotential surface and for all values of ξ^r, η^r, ζ^r, ω^r *such that*

$$V_m \xi^m = V_m \eta^m = V_m \zeta^m = V_m \omega^m = 0.$$

In these expressions $\lambda_r = V_r/(a^{mn} V_m V_n)^{1/2}$, $V_r = \partial V/\partial q^r$, *and* λ_{rs}, λ_{rst} *are covariant derivatives.*

CHAPTER VI.

DEFINITIONS OF STABILITY AND OF STEADY MOTION.

§ 6.1. *Geometrical stability.*

A definition of stability should be invariant in character—that is, independent of any particular system of co-ordinates. Further, to accord with the spirit of this paper, it should be geometrical. Such a definition I proceed to give, applying it afterwards in three special forms of peculiar dynamical significance.

Let there be a manifold of N dimensions with a co-ordinate system q^r, and let the metric be

$$(6.11) \qquad ds^2 = g_{mn}\, dq^m\, dq^n.$$

Let C and C^* be two curves whose equations are

$$(6.12) \qquad \begin{cases} (C) & q^r = \phi^r(u), \\ (C^*) & q^r = \phi^{*r}(u), \end{cases}$$

where u is a parameter. Let a correspondence be established in some definite manner between the points of these two curves. Let O and O^* be a pair of corresponding points and let Γ be the geodesic joining them. Now let us take the vector $d\phi^{*r}(u)/du$ at O^*

and propagate it parallelly along Γ. Let the result of this parallel propagation be a vector ζ^r at O. Let ξ^r denote the value of $d\phi^r(u)/du$ at O. Then the quantity Δ defined as the positive square root of

$$\Delta^2 = g_{mn}(\xi^m - \zeta^m)(\xi^n - \zeta^n) \tag{6.13}$$

is an invariant with respect to transformations of co-ordinates.

Now let us suppose that in the manifold we are given some definite system of curves defined parametrically in terms of an independent variable u; let C be a curve of the system. We shall speak of C as the *undisturbed* curve. Let a definite correspondence be set up between the points of the other curves of the system (which we shall call the *disturbed* curves) and the points of C. Let δ be any positive number and let us pick out all those disturbed curves C^* which possess a point O^* corresponding to a point O of C, such that

$$OO^* < \delta, \qquad \Delta < \delta. \tag{6.14}$$

We shall refer to every curve satisfying this condition as a *disturbed curve of order* δ.

We can now proceed to our formal definition of geometrical stability:

Definition of Geometrical Stability.—*If, being given any positive number ε, however small, a positive number δ exists such that $PP^* < \varepsilon$ for every pair of corresponding points P and P^*, P being situated on the undisturbed curve C and P^* on any disturbed curve of order δ, then the curve C is said to be stable.*

The above definition is more ambitious in point of rigour than the analytical investigations which come later. It is necessary in what succeeds to work entirely in first order effects, and it may happen that a system believed to be stable from the results of our first-order approximation is unstable in the rigorous sense. However, even if we do not make use of this definition in all its exactitude, it is desirable to be able to give a precise invariant geometrical definition.

Summed up roughly, our definition may be stated: A curve is stable when the distance between corresponding points of the curve and of an adjacent curve remains permanently small.

It will be noticed that the question of stability involves three things:—

(1) a line-element;
(2) a system of curves, each defined parametrically;
(3) a correspondence between points of the undisturbed curve and of the disturbed curves.

It is by various choices of these three things that the following types of dynamical stability are obtained.

§ 6.2. *Stability in the kinematical sense.*

The first type of dynamical stability we shall call stability in the *kinematical* sense. Here distance is measured by the kinematical line-element and the system of curves is composed of all natural trajectories under the given force system, without restriction. We shall define corresponding points as those for which t has the same value, so that simultaneous configurations correspond. Thus there is stability in the kinematical sense when the distance between simultaneous points remains permanently small.

§ 6.3. *Stability in the kinematico-statical sense.*

The second type of dynamical stability we shall call stability in the *kinematico-statical*† sense. Here again distance is measured by the kinematical line-element and the system of curves is composed of all natural trajectories under the given force system, without restriction. The correspondence between points on C and C^* is established by the condition that P should be the foot of the (geodesic) perpendicular let fall from P^* on C. Thus there is stability in the kinematico-statical sense when every point of the disturbed curve is adjacent to the undisturbed curve.

It is obvious that stability in the kinematical sense implies stability in the kinematico-statical sense, but the converse is not true.

§ 6.4. *Stability in the action sense.*

The third type of stability we shall call stability in the *action* sense. Here distance is measured by the action line-element and the system of curves consists of all natural trajectories of total energy h—that is, it consists of all the geodesics of the manifold. The correspondence between points on the curves is fixed by the condition that the arc O^*P^* should be equal to the arc OP, where O and O^* are arbitrarily selected origins on the undisturbed curve and any disturbed curve respectively. Thus the problem of stability in the action sense is that of the convergence of geodesics in Riemannian space. If two geodesics pass through adjacent points in nearly parallel directions, the distance between points on the geodesics equidistant from the respective initial points is either permanently small or not. If permanently small, there is stability. As simple examples we may quote the great circles on a sphere as illustrating stability and the straight lines on a plane as illustrating instability.

There is an important fact (easily deducible from the calculus of variations) in the case where the order of the disturbed curve is infinitesimal. It is that if O^*O is perpendicular to C, then so also is P^*P, where P and P^* are any pair of corresponding points. In considering stability in the action sense it is in general sufficient to consider only cases where O^*O is perpendicular to C; thus we may say that there is stability in the action sense when every point of the disturbed geodesic is adjacent to the undisturbed geodesic.

Perpendicularity with respect to the action line-element is equivalent to perpendicularity with respect to the kinematical line-element. Thus it appears that stability in the action sense is equivalent to stability in the kinematico-statical sense for disturbances which do not change the total energy, except in those cases where $(h-V)$ either becomes zero at some point of the undisturbed curve or tends to infinity.

† So called because we use the *kinematical* line-element, while the curves to be compared are considered *statically* as entities and not with reference to the particular motions along them.

§ 6.5. *Examples of the three types of stability.*

It should be remembered that, in the case of the motion of a particle of unit mass on a surface, the kinematical line-element is precisely the geometrical line-element of the surface as ordinarily understood, and perpendicularity of two surface directions in the sense of the kinematical or of the action line-element is equivalent to perpendicularity in the ordinary geometrical sense.

Consider the case of a particle describing an elliptical orbit under the influence of a central force varying directly as the distance. Here we clearly have stability in all three senses.

Consider the case of a particle describing an elliptical orbit under the influence of a central force varying as the inverse square of the distance. Here we have stability in the kinematico-statical sense and in the action sense, but not in the kinematical sense, since the periodic time of the disturbed orbit is in general different from that of the undisturbed orbit.

Consider the motion of a particle of unit mass on a plane under the influence of a force system derivable from a potential

(6.51) $$V = -x + \tfrac{1}{2}y^2.$$

Writing down the equations of motions and solving, we get

(6.52) $$\begin{cases} x = \tfrac{1}{2}t^2 + At + B, \\ y = C\sin(t + D), \end{cases}$$

where A, B, C and D are constants of integration. Let the undisturbed motion be

(6.53) $$\begin{cases} x = \tfrac{1}{2}t^2 + t, \\ y = 0. \end{cases}$$

The motion is clearly unstable in the kinematical sense. In considering stability in the kinematico-statical sense, the distance between corresponding points is

(6.54) $$PP^* = y = C\sin(t + D).$$

Thus there is stability in the kinematico-statical sense. To discuss stability in the action sense, let us take the initial point O at the origin of co-ordinates and the initial point O^* on the y-axis. Then, the disturbance being infinitesimal, the (action) distance between corresponding points is

(6.55) $$PP^* = (h - V)^{\frac{1}{2}}\, y = 2^{-\frac{1}{2}}(t + 1)\, C\sin(t + D).$$

Thus there is instability in the action sense. This example illustrates how such instability occurs when the kinetic energy tends to infinity in the undisturbed motion.

Consider the motion of a particle in a parallel uniform field of force. Here we have instability in all three senses.

§ 6.6. *Steady motions in the kinematical sense.*

A somewhat incomplete definition of steady motion has been given by ROUTH.† Another definition has been given by WHITTAKER,‡ which may be stated as follows:

A motion is steady when all the components of the velocity vector are constant throughout the motion, those corresponding to non-ignorable co-ordinates being zero.

We shall refer to such a motion as *steady in the ignorable sense.* While such a definition is undoubtedly very convenient for the discussion of stability by means of the Hamiltonian equations, it appears to be open to criticism on two grounds. First, it pre-supposes that we are already in possession of one of those co-ordinate systems for which there are ignorable co-ordinates, and, secondly, it defines steadiness of motion in terms of the properties of the whole manifold of configurations.

The definition which follows provides tests by which it may be determined directly by calculation whether or not a given motion is steady, the motion being defined by equations

(6.61) $$q^r = q^r(t)$$

for a perfectly general co-ordinate system and without any reference to ignorable co-ordinates. The definition will probably appear artificial and the reason for its adoption will only become clear when we come to discuss the question of stability analytically in the succeeding chapters. It will then be seen that the definition appears to satisfy the fundamental idea of ROUTH.

Let the metric be the kinematical line-element and let the notation for the normals to the trajectory be as in § 2.7. Let us write

(6.62) $$K_{(P,Q)} = G_{mnst}\lambda^m_{(P)}\lambda^n_{(0)}\lambda^s_{(Q)}\lambda^t_{(0)}, \quad (P, Q = 0, 1, \ldots, N-1),$$

where G_{mnst} is the curvature tensor of the manifold of configurations. We note at once from the well-known properties of the curvature tensor that $K_{(P,Q)}$ is symmetric in P and Q and that

(6.621) $$K_{(P,0)} = K_{(0,P)} = 0, \qquad (P = 0, 1, \ldots, N-1).$$

We may observe that $K_{(P,P)}$ is the Riemannian curvature of the manifold of configurations corresponding to the two-space element defined by the tangent to the trajectory and its Pth normal. Let us write

(6.63) $$W_{(P,Q)} = Q_{mn}\lambda^m_{(P)}\lambda^n_{(Q)}, \qquad (P, Q = 0, 1, \ldots, N-1),$$

where Q_{mn} is the covariant derivative of the covariant force vector. We note that $W_{(P,Q)}$ is not in general symmetric in P and Q; it is, however, symmetric when a potential exists, for then we have

(6.64) $$W_{(P,Q)} = -V_{mn}\lambda^m_{(P)}\lambda^n_{(Q)} = -V_{nm}\lambda^m_{(P)}\lambda^n_{(Q)} = W_{(Q,P)}, \qquad (P, Q = 0, 1, \ldots, N-1).$$

† *A Treatise on the Stability of Motion* (1877), 2.

‡ *Loc. cit.*, 193.

We may also observe that, when a potential exists, $W_{(P,P)}$ is equal to the value of $-d^2V/ds^2$ calculated along a geodesic drawn from the point in the direction of the Pth normal. The quantities $K_{(P,Q)}$ and $W_{(P,Q)}$ are, of course, invariant with respect to transformations of co-ordinates.

We shall now proceed to our definition :

Definition (K).—*A natural motion is said to be steady in the kinematical sense when all along the trajectory—*

(1) *the velocity* v *is constant ;*

(2) *all the curvatures* $\kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$ *are constant* ;

(3) *all the quantities* $K_{(P,Q)}$ $(P, Q = 1, 2, \ldots, N-1)$ *are constant* ;

(4) *all the quantities* $W_{(P,Q)}$ $(P, Q = 1, 2, \ldots, N-1)$ *are constant.*

This is the definition of steady motion for discussions involving the kinematical line-element, that is to say, for the treatment of the question of stability in the kinematical and kinematico-statical senses.

If none of the curvatures $\kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$ vanish, all the quantities occurring in the above definition are uniquely defined. If, however, several of the curvatures vanish, some of the normals are no longer uniquely determined, and we must show that our definition provides a test of steadiness independent of any arbitrariness in the choice of normals. Let us suppose that the Mth curvature vanishes, all normals of lower order being uniquely defined by (2.71). Let $\lambda^r_{(M)}, \lambda^r_{(M+1)}, \ldots, \lambda^r_{(N-1)}$ and $\lambda^{*r}_{(M)}, \lambda^{*r}_{(M+1)}, \ldots, \lambda^{*r}_{(N-1)}$, be two sets of unit vectors chosen to represent the Mth, $(M+1)$th, $\ldots$, $(N-1)$th normals. If we write

(6.65) $$\lambda^{*r}_{(P)} = \lambda^r_{(P)}, \qquad (P = 1, 2, \ldots, M-1),$$

(6.651) $$K^*_{(P,Q)} = G_{mnst}\lambda^{*m}_{(P)}\lambda^n_{(0)}\lambda^{*s}_{(Q)}\lambda^t_{(0)}$$

(6.652) $$W^*_{(P,Q)} = Q_{mn}\lambda^{*m}_{(P)}\lambda^{*n}_{(Q)}$$

$$(P, Q = 1, 2, \ldots, N-1),$$

we shall prove that

(6.66) $$K_{(P,Q)} = \textit{constant}, \qquad (P, Q = 1, 2, \ldots, N-1),$$

imply

(6.661) $$K^*_{(P,Q)} = \textit{constant}, \qquad (P, Q = 1, 2, \ldots, N-1),$$

and that

(6.67) $$W_{(P,Q)} = \textit{constant}, \qquad (P, Q = 1, 2, \ldots, N-1),$$

imply

(6.671) $$W^*_{(P,Q)} = \textit{constant}, \qquad (P, Q = 1, 2, \ldots, N-1),$$

where " constant " means " constant along the trajectory."

The condition of perpendicularity with respect to all the normals of order less than M and to the tangent restricts the normals of order equal to or greater than M to an element of $(N - M)$ dimensions at every point of the trajectory. Thus we may write

(6.68) $$\lambda^{*r}_{(P)} = \beta_{(P, M)}\lambda^r_{(M)} + \beta_{(P, M+1)}\lambda^r_{(M+1)} + \dots + \beta_{(P, N-1)}\lambda^r_{(N-1)}, \quad (P = M, M+1, \dots, N-1).$$

Taking the contravariant space derivative and remembering that all the vectors occurring in (6.68) are propagated parallelly along the trajectory, we have

(6.681) $$\beta'_{(P, M)}\lambda^r_{(M)} + \beta'_{(P, M+1)}\lambda^r_{(M+1)} + \dots + \beta'_{(P, N-1)}\lambda^r_{(N-1)} = 0, \quad (P = M, M+1, \dots, N-1).$$

If we multiply by $a_{rs}\lambda^s_{(Q)}$, where Q is one of the numbers $M, M+1, \dots, N-1$, and sum as indicated, we obtain

(6.682) $$\beta'_{(P, Q)} = 0, \qquad (P, Q = M, M+1, \dots, N-1),$$

by virtue of the condition of mutual perpendicularity of the normals. Thus all the β co-efficients occurring in (6.68) are constants along the trajectory.

Now if we put

(6.683) $$\beta_{(P, Q)} = \left\{\begin{array}{l} 1 \text{ for } P = Q \\ 0 \text{ for } P \neq Q \end{array}\right\} (P, Q = 1, 2, \dots, M-1),$$

and

(6.684) $$\beta_{(P, Q)} = \beta_{(Q, P)} = 0, \quad (P = 1, 2, \dots, M-1; \quad Q = M, M+1, \dots, N-1),$$

we may write

(6.685) $$\lambda^{*r}_{(P)} = \sum_{R=1}^{N-1} \beta_{(P, R)}\lambda^r_{(R)}, \qquad (P = 1, 2, \dots, N-1).$$

Thus we have

(6.686) $$K^*_{(P, Q)} = \sum_{R=1}^{N-1}\sum_{S=1}^{N-1} G_{mnst}\beta_{(P, R)}\lambda^m_{(R)}\lambda^n_{(0)}\beta_{(Q, S)}\lambda^s_{(S)}\lambda^t_{(0)},$$
$$= \sum_{R=1}^{N-1}\sum_{S=1}^{N-1} \beta_{(P, R)}\beta_{(Q, S)}K_{(R, S)}, \qquad (P, Q = 1, 2, \dots, N-1).$$

Therefore, since the β's are all constant, (6.66) implies (6.661). Similarly (6.67) implies (6.671). Our definition of steady motion is therefore free from all ambiguity in the case of vanishing curvatures.

We shall now show that any motion which is steady in the ignorable sense is also steady in the sense of our definition, provided that none of the $(N - 2)$ curvatures $\kappa_{(1)}, \kappa_{(2)}, \dots, \kappa_{(N-2)}$ vanish.

Let us first consider the case where none of the $(N - 1)$ curvatures vanish. Since only the ignorable co-ordinates change and all their rates of change are constant, it follows that any expression involving only the components of the fundamental tensor a_{mn} and their partial derivatives with respect to the co-ordinates, the components of velocity and the potential, remains constant throughout the motion. Thus T is constant and therefore v is constant; hence ds/dt is constant. Now, when we examine the equations (2.71), we see that $\lambda^r_{(1)}, \lambda^r_{(2)}, \ldots, \lambda^r_{(N-1)}, \kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$ are defined as functions of the components of the fundamental tensor a_{mn} and their partial derivatives with respect to the co-ordinates and of $\lambda^r_{(0)}$ and their derivatives with respect to s. But

(6.69) $$\lambda^r_{(0)} = q^{r'} = \dot{q}^r \frac{dt}{ds},$$

which are constant along the trajectory. Hence, since a_{mn} and their partial derivatives are also constant along the trajectory, we see that $\lambda^r_{(1)}, \lambda^r_{(2)}, \ldots, \lambda^r_{(N-1)}, \kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$, are all constant along the trajectory. Hence we see at once that $K_{(P,Q)}$ are constant. Also V_{mn} are evidently constant, since they are independent of the ignorable co-ordinates, and thus $W_{(P,Q)}$ are constant. Thus we find that steadiness in the ignorable sense implies steadiness in the sense of our definition, provided that none of the curvatures $\kappa_{(1)}, \kappa_{(2)}, \ldots, \kappa_{(N-1)}$ vanish.

Let us now consider the case where just one of the curvatures $(\kappa_{(N-1)})$ vanishes throughout the motion. As pointed out in § 2.7, the $(N-1)$th normal is then uniquely defined (except with respect to sense) by the equations

(6.691) $$a_{mn}\lambda^m_{(N-1)}\lambda^n_{(P)} = 0, \qquad (P = 0, 1, \ldots, N-2),$$

and

(6.692) $$a_{mn}\lambda^m_{(N-1)}\lambda^n_{(N-1)} = 1.$$

But when the motion is steady in the ignorable sense, a_{mn} and $\lambda^r_{(P)}$ $(P = 0, 1, \ldots, N-2)$ are constant along the trajectory, and therefore $\lambda^r_{(N-1)}$ are also constant along the trajectory, from which it follows at once that the motion is steady in the kinematical sense.

In the case where several curvatures vanish, it does not appear to be true in general that steadiness in the ignorable sense implies steadiness in the sense of our definition.

§ 6.7. *Steady curves in the action sense.*

In dealing with the action line-element, we should remember that the theory developed is precisely the geometrical theory of a Riemannian manifold. A steady motion will correspond to a geodesic of the manifold having special properties and, to stress the geometrical character, we shall speak of a *steady curve* instead of a steady motion.

Let $\lambda^r_{(0)}$ denote the unit vector tangent to a geodesic C and let $\lambda^r_{(1)}, \lambda^r_{(2)}, \ldots, \lambda^r_{(N-1)}$ denote any $(N-1)$ mutually perpendicular unit vectors perpendicular to C and propagated parallelly along C. It is to be remembered that if we take any set of unit vectors at a point of a geodesic and propagate them parallelly along the geodesic, the angles between the vectors and the angles between the vectors and the tangent to the geodesic all remain constant.† Now let

(6.71) $$K_{(P,Q)} = G_{mnst}\lambda^m_{(P)}\lambda^n_{(0)}\lambda^s_{(Q)}\lambda^t_{(0)}, \qquad (P, Q = 0, 1, \ldots, N-1).$$

† *Cf.* BIANCHI, *loc. cit.*, 794.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Clearly $K_{(P, Q)}$ is symmetric in P and Q and $K_{(P, P)}$ is equal to the Riemannian curvature of the manifold of configurations corresponding to the two-space element defined by the tangent and the direction $\lambda^r_{(P)}$.

We shall now define a steady curve :

DEFINITION (A).—*A steady curve in the action sense is a geodesic along which all the quantities*

$$K_{(P, Q)}, \qquad (P, Q = 1, 2, \ldots, N-1),$$

are constant.

This is the definition of a steady motion to be adopted in discussing the question of stability in the action sense.

We shall not delay to prove that our definition has a unique significance independent of the choice of the particular set of mutually perpendicular normals $\lambda^r_{(P)}$ $(P = 1, 2, \ldots, N-1)$. The mode of proof is as in § 6.6.

It does not appear to be generally true that a motion which is steady in the ignorable sense (*cf.* § 6.6.) is necessarily steady in the sense of the above definition.

CHAPTER VII.

STABILITY IN THE KINEMATICAL SENSE.

§ 7.1. *Equations for the components of the disturbance vector.*

It is in the analytical investigations in connection with stability of motion that the use of the tensorial notation becomes of greatest importance. The appearance of the Riemannian curvature tensor in the course of the analysis makes it difficult to believe that similar results could be obtained without the use of this method.

The equations of an undisturbed natural trajectory C are

$$\ddot{q}^r + \begin{Bmatrix} mn \\ r \end{Bmatrix} \dot{q}^m \dot{q}^n = Q^r. \tag{7.11}$$

Let q^r be the co-ordinates of a point P of C and $(q^r + \eta^r)$ the co-ordinates of the corresponding (simultaneous) point P^* of the disturbed natural trajectory C^*, η^r being infinitesimal. We shall call the vector η^r the *disturbance vector.* The condition for stability in the kinematical sense is that η (the magnitude of the vector η^r) should remain permanently small.

Let us substitute $(q^r + \eta^r)$ in (7.11), since C^* is a natural trajectory, and obtain

$$\ddot{q}^r + \ddot{\eta}^r + \begin{Bmatrix} mn \\ r \end{Bmatrix}^* (\dot{q}^m + \dot{\eta}^m)(\dot{q}^n + \dot{\eta}^n) - (Q^r)^* = 0, \tag{7.12}$$

where the asterisk indicates quantities to be calculated at P^*. Expanding these quantities and retaining only first powers of small quantities, we have, after making use of (7.11),

$$\ddot{\eta}^r + 2 \begin{Bmatrix} m\, n \\ r \end{Bmatrix} \dot{\eta}^m \dot{q}^n + \frac{\partial}{\partial q^s} \begin{Bmatrix} m\, n \\ r \end{Bmatrix} \eta^s \dot{q}^m \dot{q}^n - \frac{\partial Q^r}{\partial q^s} \eta^s = 0. \tag{7.121}$$

But by the definition of the contravariant time-flux (2.22) we have

(7.13) $$\hat{\eta}^r = \dot{\eta}^r + \begin{Bmatrix} m\,n \\ r \end{Bmatrix} \eta^m \dot{q}^n,$$

and hence

(7.131) $$\hat{\hat{\eta}}^r = \frac{d}{dt}\left(\dot{\eta}^r + \begin{Bmatrix} m\,n \\ r \end{Bmatrix} \eta^m \dot{q}^n\right) + \begin{Bmatrix} s\,t \\ r \end{Bmatrix}\left(\dot{\eta}^s + \begin{Bmatrix} m\,n \\ s \end{Bmatrix} \eta^m \dot{q}^n\right)\dot{q}^t$$
$$= \ddot{\eta}^r + 2\begin{Bmatrix} m\,n \\ r \end{Bmatrix}\dot{\eta}^m\dot{q}^n + \frac{\partial}{\partial q^s}\begin{Bmatrix} m\,n \\ r \end{Bmatrix}\eta^m\dot{q}^n\dot{q}^s + \begin{Bmatrix} m\,n \\ r \end{Bmatrix}\eta^m\ddot{q}^n + \begin{Bmatrix} s\,t \\ r \end{Bmatrix}\begin{Bmatrix} m\,n \\ s \end{Bmatrix}\eta^m\dot{q}^n\dot{q}^t.$$

Substituting for $\ddot{q}^n$ from (7.11) we find (after making the necessary changes of indices)

(7.132) $$\ddot{\eta}^r + 2\begin{Bmatrix} m\,n \\ r \end{Bmatrix}\dot{\eta}^m\dot{q}^n = \hat{\hat{\eta}}^r - \left(\frac{\partial}{\partial q^n}\begin{Bmatrix} m\,s \\ r \end{Bmatrix} - \begin{Bmatrix} s\,t \\ r \end{Bmatrix}\begin{Bmatrix} m\,n \\ t \end{Bmatrix} + \begin{Bmatrix} n\,t \\ r \end{Bmatrix}\begin{Bmatrix} m\,s \\ t \end{Bmatrix}\right)\eta^s\dot{q}^m\dot{q}^n$$
$$- \begin{Bmatrix} m\,n \\ r \end{Bmatrix}\eta^m Q^n.$$

Substitution in (7.121) gives

(7.14) $$\hat{\hat{\eta}}^r + G^r_{msn}\eta^s\dot{q}^m\dot{q}^n - Q^r_s\eta^s = 0,$$

where

(7.141) $$G^r_{msn} = \frac{\partial}{\partial q^s}\begin{Bmatrix} m\,n \\ r \end{Bmatrix} - \frac{\partial}{\partial q^n}\begin{Bmatrix} m\,s \\ r \end{Bmatrix} + \begin{Bmatrix} m\,n \\ t \end{Bmatrix}\begin{Bmatrix} s\,t \\ r \end{Bmatrix} - \begin{Bmatrix} m\,s \\ t \end{Bmatrix}\begin{Bmatrix} n\,t \\ r \end{Bmatrix},$$

the mixed curvature tensor of the manifold of configurations for the kinematical line-element, and

(7.142) $$Q^r_s = \frac{\partial Q^r}{\partial q^s} + \begin{Bmatrix} s\,n \\ r \end{Bmatrix} Q^n,$$

the covariant derivative of the force vector Q^r. It will generally be more convenient to apply (7.14) in the covariant form

(7.15) $$a_{rs}\hat{\hat{\eta}}^s + G_{rmsn}\dot{q}^m\eta^s\dot{q}^n - Q_{rs}\eta^s = 0.$$

Equation (7.14) or (7.15) may be called *the tensorial equation for the disturbance vector in the kinematical sense.*

§ 7.2. *Equation for the magnitude of the disturbance vector.*

Let us introduce the *unit disturbance vector* μ^r co-directional with η^r, so that

(7.21) $$\eta^r = \eta\mu^r,$$

and

(7.211) $$a_{mn}\mu^m\mu^n = 1.$$

Then

(7.22) $$\hat{\eta}^r = \dot{\eta}\mu^r + \eta\hat{\mu}^r,$$

and

(7.221) $$\hat{\hat{\eta}}^r = \ddot{\eta}\mu^r + 2\dot{\eta}\hat{\mu}^r + \eta\hat{\hat{\mu}}^r.$$

Therefore

(7.222) $$a_{rt}\hat{\hat{\eta}}^r\mu^t = \ddot{\eta} + 2\dot{\eta}a_{rt}\hat{\mu}^r\mu^t + \eta a_{rt}\hat{\hat{\mu}}^r\mu^t.$$

But from (7.211) we obtain

(7.23) $$a_{mn}\hat{\mu}^m\mu^n = 0,$$

and hence

(7.231) $$a_{mn}\hat{\hat{\mu}}^m\mu^n + a_{mn}\hat{\mu}^m\hat{\mu}^n = 0.$$

Thus (7.222) may be written

(7.24) $$a_{rt}\hat{\hat{\eta}}^r\mu^t = \ddot{\eta} - \eta a_{rt}\hat{\mu}^r\hat{\mu}^t.$$

Now if we multiply (7.15) by μ^r and sum as indicated, we have, by (7.24),

(7.25), $$\ddot{\eta} - \eta a_{rt}\hat{\mu}^r\hat{\mu}^t + G_{rmsn}\mu^r\dot{q}^m\eta^s\dot{q}^n - Q_{rs}\mu^r\eta^s = 0,$$

which may be written

(7.26) $$\ddot{\eta} + \eta\,(G_{mnst}\mu^m\dot{q}^n\mu^s\dot{q}^t - \hat{\mu}^2 - Q_{mn}\mu^m\mu^n) = 0,$$

where $\hat{\mu}$ is the magnitude of the vector $\hat{\mu}^r$. Equation (7.26) may be called *the invariant equation for the magnitude of the disturbance in the kinematical sense.*

The invariant $G_{mnst}\mu^m\dot{q}^n\mu^s\dot{q}^t$ is equal to the Riemannian curvature of the manifold of configurations corresponding to the directions μ^r and $\dot{q}^r$, multiplied by a positive factor ; hence we may state the following result :

THEOREM XXVI (K).—*If the Riemannian curvature of the manifold of configurations corresponding to every two-space element containing the direction of the given trajectory is negative or zero, and if $Q_{mn}x^mx^n$ is positive or zero for arbitrary values of x^r at all points of the trajectory, then the motion is unstable in the kinematical sense.*

§ 7.3. *The integral of energy.*

In the case where the force system is conservative we have also the integral of energy, which may be written

(7.31) $$T^* + V^* = T + V + \delta h,$$

where δh is the excess of the total energy in the disturbed motion over the total energy in the undisturbed motion.

But

(7.32) $$2T^* = \left(a_{mn} + \frac{\partial a_{mn}}{\partial q^s}\eta^s\right)(\dot{q}^m + \dot{\eta}^m)(\dot{q}^n + \dot{\eta}^n).$$

which easily reduces to

(7.321) $$T^* = T + a_{mn}\dot{q}^m\hat{\eta}^n.$$

Also

(7.33) $$V^* = V + V_m\eta^m.$$

Thus the integral of energy is

$$(7.34) \qquad a_{mn}\dot{q}^m\hat{\eta}^n + V_m\eta^m = \delta h,$$

or, in alternative form, by substitution from (7.22) and (7.21),

$$(7.35) \qquad \dot{\eta}a_{mn}\dot{q}^m\mu^n + \eta\,(a_{mn}\dot{q}^m\hat{\mu}^n + V_m\mu^m) = \delta h.$$

§ 7.4. *The case of a conservative system with two degrees of freedom.*

The theory of the vibrations of a conservative system with two degrees of freedom finds its most obvious application in the study of the motion of a particle on a plane or curved surface in Euclidean space of three dimensions. It must not be forgotten, however, that the theory as here developed is much more general, embracing as it does problems of rigid dynamics also. It is true that the results of this and the next sections may be regarded as particular cases of those developed in § 7.6, but the relative simplicity and importance of the cases of two and three degrees of freedom seem to warrant independent treatments.

Let λ^r be the unit vector tangent to the undisturbed trajectory and let ν^r be the unit vector of the principal normal, *i.e.*, the unit vector drawn normal to the curve out from the concave side. Equations (2.711) are applicable, but will be written in the more convenient form

$$(7.41) \qquad \hat{\lambda}^r = \omega\nu^r, \qquad \hat{\nu}^r = -\omega\lambda^r, \qquad \omega = v\kappa,$$

where v is the velocity. The quantity ω may be called the *angular velocity* of the undisturbed motion. We at once derive

$$(7.411) \qquad \hat{\hat{\lambda}}^r = -\omega^2\lambda^r + \dot{\omega}\nu^r, \qquad \hat{\hat{\nu}}^r = -\omega^2\nu^r - \dot{\omega}\lambda^r.$$

The force system being conservative, we have

$$(7.42) \qquad Q_r = -V_r, \qquad Q_{rs} = -V_{rs} = -V_{sr} = Q_{sr}.$$

If we equate the normal acceleration to the normal force component and the tangential acceleration to the tangential force component, we obtain

$$(7.421) \qquad v\omega = v^2\kappa = -V_m\nu^m, \qquad \dot{v} = -V_m\lambda^m,$$

and hence, by differentiation of the first of these equations with respect to the time and substitution from the second, we easily find

$$(7.422) \qquad 2\dot{v}\omega + v\dot{\omega} = -vV_{mn}\nu^m\lambda^n.$$

Now let us write

$$(7.43) \qquad \eta^r = \alpha\lambda^r + \beta\nu^r,$$

so that α and β are the components of the disturbance vector in the directions of the tangent and the normal of the trajectory respectively. On differentiation we obtain

(7.431) $$\hat{\eta}^r = \alpha\hat{\lambda}^r + \dot{\alpha}\lambda^r + \beta\hat{\nu}^r + \dot{\beta}\nu^r,$$
$$= \lambda^r(\dot{\alpha} - \omega\beta) + \nu^r(\dot{\beta} + \omega\alpha),$$

and

(7.432) $$\hat{\hat{\eta}}^r = \lambda^r(\ddot{\alpha} - \omega^2\alpha - 2\omega\dot{\beta} - \dot{\omega}\beta) + \nu^r(\ddot{\beta} - \omega^2\beta + 2\omega\dot{\alpha} + \dot{\omega}\alpha).$$

Thus

(7.433) $$a_{rs}\hat{\hat{\eta}}^r\nu^s = \ddot{\beta} - \omega^2\beta + 2\omega\dot{\alpha} + \dot{\omega}\alpha.$$

For the solution of the problem of vibrations we shall employ the equation (7.15) for the components of the disturbance vector and the integral of energy (7.34). From the former we have

7.44) $$a_{rs}\hat{\hat{\eta}}^r\nu^s = -G_{rmsn}\nu^r\dot{q}^m\eta^s\dot{q}^n - V_{rs}\nu^r\eta^s.$$

If we substitute from (7.43) and remember the skew-symmetric property of the curvature tensor, the first term on the right-hand side becomes

(7.441) $$-\beta G_{rmsn}\nu^r\dot{q}^m\nu^s\dot{q}^n.$$

Thus, if K denotes the Gaussian curvature of the manifold of configurations, we may write (7.44) in the form

(7.442) $$a_{rs}\hat{\hat{\eta}}^r\nu^s = -\beta v^2 K - \alpha V_{mn}\nu^m\lambda^n - \beta V_{mn}\nu^m\nu^n.$$

Then, comparing this equation with (7.433), we have the equation of motion

(7.443) $$\ddot{\beta} - \omega^2\beta + 2\omega\dot{\alpha} + \dot{\omega}\alpha = -\beta v^2 K - \alpha V_{mn}\nu^m\lambda^n - \beta V_{mn}\nu^m\nu^n,$$

or, by (7.422),

(7.444) $$v\ddot{\beta} + v\beta(v^2K + V_{mn}\nu^m\nu^n - \omega^2) + 2\omega(v\dot{\alpha} - \dot{v}\alpha) = 0.$$

The equation of energy (7.34) becomes, after use of (7.431),

(7.45) $$v(\dot{\alpha} - \omega\beta) + \alpha V_m\lambda^m + \beta V_m\nu^m = \delta h,$$

or, by (7.421),

(7.451) $$v\dot{\alpha} - \dot{v}\alpha = 2v\omega\beta + \delta h.$$

In equations (7.444) and (7.451) everything (except α and β and their derivatives) may be considered as given functions of the time t when the undisturbed trajectory is given. These two equations contain the solution to the problem. If we eliminate $(v\dot{\alpha} - \dot{v}\alpha)$, we obtain an equation for β only

(7.452) $$v\ddot{\beta} + v\beta(v^2K + V_{mn}\nu^m\nu^n + 3\omega^2) + 2\omega\delta h = 0,$$

or

(7.453) $$\ddot{\beta} + \beta(v^2K + V_{mn}\nu^m\nu^n + 3v^2\kappa^2) + 2\kappa\delta h = 0.$$

The value of α is given in terms of β by (7.451) and we find

$$\alpha = 2v\int_0^t \kappa\beta dt + v\delta h\int_0^t \frac{dt}{v^2} + Av, \tag{7.454}$$

where A is an infinitesimal constant equal to the initial value of α/v. Since, by (7.43),

$$\eta^2 = \alpha^2 + \beta^2, \tag{7.46}$$

it follows that for stability in the kinematical sense it is necessary and sufficient that both α and β should remain permanently small. We may state the result :

THEOREM XXVII (K).—*In order that the motion of a holonomic conservative system with two degrees of freedom may be stable in the kinematical sense, it is necessary and sufficient that the value of* β *as solution of the differential equation* (7.453) *and the value of* α *as given by* (7.454) *should be permanently small, for all arbitrary infinitesimal values of the constants* δh *and* A.

It should be noted that, in the case where β (as the solution of (7.453)) turns out to be permanently small, but where α (as given by (7.454)) does not remain permanently small, we cannot deduce that β (as defined by (7.43)) remains permanently small, since equation (7.453) has been obtained on the assumption that α as well as β is small. WHITTAKER† discusses the stability of the orbit of a particle in a plane. His argument appears open to criticism on the above grounds. The coefficient of β in (7.453) can be identified immediately with the coefficient of stability, K being zero in the case of motion in a plane. It will be seen in § 8.3 that a positive coefficient of stability ensures stability in the kinematico-statical sense for disturbances that do not change the total energy,‡ but it does not ensure stability in the kinematical sense, as the argument of Whittaker seems to imply.

We shall call the quantity

$$v^2K + V_{mn}\nu^m\nu^n + 3v^2\kappa^2 \tag{7.461}$$

the (generalised) *coefficient of stability*.

If the total energy is not changed by the disturbance, δh is zero and STURM's theorem§ is applicable. We may state the following result :

THEOREM XXVIII (K).—*If all along a natural trajectory of a holonomic conservative system with two degrees of freedom the coefficient of stability is positive, then the motion is stable in the kinematical sense for all disturbances which do not change the total energy, provided that the value of* α *given by*

$$\alpha = 2v\int_0^t \kappa\beta\, dt + Av \tag{7.462}$$

remains permanently small for all arbitrary infinitesimal values of the constant A.

† *Loc. cit.*, 395.

‡ This result for a particle orbit in a plane follows directly from equation (12) of THOMSON and TAIT, *Natural Philosophy*, 1 (1879), 428. Their argument is free from the objectionable feature to which we have alluded.

§ *Cf.* DARBOUX, *Théorie générale des surfaces*, Pt. 3 (1894), 102.

When the motion is steady, the coefficient of stability, the curvature (κ) and the velocity (v) are all constant along the trajectory (*cf.* § 6.6). Thus, if we write c^2 for the coefficient of stability, assuming it to be positive, we have as the solution of (7.453)

$$\beta = B\cos ct + C\sin ct - \frac{2\kappa\delta h}{c^2}. \tag{7.47}$$

Hence

$$\alpha = \frac{2v\kappa}{c}(B\sin ct - C\cos ct) + \frac{t\delta h}{vc^2}(c^2 - 4v^2\kappa^2) + constant. \tag{7.471}$$

Thus, taking the general case where δh is not zero, there is stability or instability according as $(c^2 - 4v^2\kappa^2)$ is zero or not zero; we may state the result:

THEOREM XXIX (K).—*In order that a steady motion of a holonomic conservative system with two degrees of freedom may be stable in the kinematical sense, it is necessary and sufficient that*

$$v^2K + V_{mn}\nu^m\nu^n - v^2\kappa^2 = 0. \tag{7.472}$$

The periodic time of a stable vibration is $\pi/v\kappa$.

As a simple application of this theorem, consider the motion of a particle in a plane under a central force whose magnitude depends only on the distance from the centre. For any radius r there exists a circular motion, which will, of course, be steady. The preceding theorem gives as condition for stability

$$\frac{d^2V}{dr^2} = \frac{v^2}{r^2}, \tag{7.473}$$

and the equilibrium equation is

$$\frac{dV}{dr} = \frac{v^2}{r}. \tag{7.474}$$

If all these circular orbits are stable, we may eliminate v and solve the resulting equation, obtaining

$$V = kr^2 + k', \tag{7.475}$$

where k and k' are constants. Thus the only law of central force depending on the distance for which all circular motions are stable in the kinematical sense is that of the direct distance.

§ 7.5. *The case of a conservative system with three degrees of freedom.*

As important types of motion to which the following theory is applicable we may mention the motion of a particle in three-dimensional Euclidean space and the motion of a rigid body about its centre of gravity or about a fixed point.

Let λ^r be the unit vector tangent to the undisturbed trajectory and ν^r and ρ^r unit vectors co-directional with the first and second normals respectively. The first and second curvatures being denoted by κ and σ respectively, formulæ (2.712) are applicable, but we shall write them in the more convenient form

(7.51)
$$\begin{cases} \hat{\lambda}^r = \omega \nu^r, \\ \hat{\nu}^r = \Omega \rho^r - \omega \lambda^r, \\ \hat{\rho}^r = -\Omega \nu^r. \end{cases} \qquad \omega = v\kappa, \quad \Omega = v\sigma,$$

The quantities ω and Ω may be called respectively the first and second angular velocities of the undisturbed motion. We at once derive

(7.511)
$$\begin{cases} \hat{\hat{\lambda}}^r = -\omega^2 \lambda^r + \dot{\omega} \nu^r + \omega \Omega \rho^r, \\ \hat{\hat{\nu}}^r = -\dot{\omega} \lambda^r - (\omega^2 + \Omega^2) \nu^r + \dot{\Omega} \rho^r, \\ \hat{\hat{\rho}}^r = \omega \Omega \lambda^r - \dot{\Omega} \nu^r - \Omega^2 \rho^r. \end{cases}$$

By resolution of the force vector along the normals and tangent we have

(7.52)
$$0 = -V_m \rho^m, \qquad v\omega = -V_m \nu^m, \qquad \dot{v} = -V_m \lambda^m,$$

and by differentiation of the second of these equations with respect to the time we obtain without difficulty

(7.521)
$$2\dot{v}\omega + v\dot{\omega} = -vV_{mn} \nu^m \lambda^n,$$

the same result as in § 7.4. Also, by differentiation of the first of (7.52) we find

(7.522)
$$V_{mn} \rho^m \lambda^n = -\omega \Omega.$$

Now let us put

(7.53)
$$\eta^r = \alpha \lambda^r + \beta \nu^r + \gamma \rho^r,$$

so that α, β and γ are the components of the disturbance in the directions of the tangent, first normal and second normal respectively. We at once obtain

(7.531)
$$\hat{\eta}^r = \dot{\alpha} \lambda^r + \dot{\beta} \nu^r + \dot{\gamma} \rho^r + \alpha \hat{\lambda}^r + \beta \hat{\nu}^r + \gamma \hat{\rho}^r,$$

and

(7.532)
$$\hat{\hat{\eta}}^r = \ddot{\alpha} \lambda^r + \ddot{\beta} \nu^r + \ddot{\gamma} \rho^r + 2\dot{\alpha} \hat{\lambda}^r + 2\dot{\beta} \hat{\nu}^r + 2\dot{\gamma} \hat{\rho}^r + \alpha \hat{\hat{\lambda}}^r + \beta \hat{\hat{\nu}}^r + \gamma \hat{\hat{\rho}}^r.$$

Hence

(7.533)
$$a_{rs} \hat{\hat{\eta}}^r \nu^s = \ddot{\beta} - \beta(\omega^2 + \Omega^2) - 2\dot{\gamma}\Omega - \gamma\dot{\Omega} + 2\dot{\alpha}\omega + \alpha\dot{\omega},$$

and

(7.534)
$$a_{rs} \hat{\hat{\eta}}^r \rho^s = \ddot{\gamma} - \gamma \Omega^2 + 2\dot{\beta}\Omega + \beta\dot{\Omega} + \alpha\omega\Omega.$$

Now from the equation (7.15) for the components of the disturbance vector we have

(7.54)
$$a_{rs} \hat{\hat{\eta}}^r \nu^s = -G_{rmsn} \nu^r \dot{q}^m (\alpha \lambda^s + \beta \nu^s + \gamma \rho^s) \dot{q}^n - V_{rs} \nu^r (\alpha \lambda^s + \beta \nu^s + \gamma \rho^s),$$

or, with the notation of § 6.6,

(7.541) $$a_{rs}\hat{\hat{\eta}}^r\nu^s = -v^2\beta K_{(1,1)} - v^2\gamma K_{(1,2)} - \alpha V_{rs}\nu^r\lambda^s - \beta V_{rs}\nu^r\nu^s - \gamma V_{rs}\nu^r\rho^s,$$

and similarly

(7.542) $$a_{rs}\hat{\hat{\eta}}^r\rho^s = -v^2\beta K_{(1,2)} - v^2\gamma K_{(2,2)} - \alpha V_{rs}\rho^r\lambda^s - \beta V_{rs}\rho^r\nu^s - \gamma V_{rs}\rho^r\rho^s.$$

Comparing these equations with (7.533) and (7.534) and using (7.521) and (7.522), we obtain the equations

(7.543) $$v\ddot{\beta} + v\beta\,[v^2K_{(1,1)} + V_{mn}\nu^m\nu^n - (\omega^2 + \Omega^2)] - 2v\dot{\gamma}\Omega + v\gamma\,[v^2K_{(1,2)} + V_{mn}\nu^m\rho^n - \dot{\Omega}] + 2\omega\,(v\dot{\alpha} - \dot{v}\alpha) = 0,$$

and

(7.544) $$\ddot{\gamma} + \gamma\,[v^2K_{(2,2)} + V_{mn}\rho^m\rho^n - \Omega^2] + 2\dot{\beta}\Omega + \beta\,[v^2K_{(1,2)} + V_{mn}\nu^m\rho^n + \dot{\Omega}] = 0.$$

These two equations and the integral of energy (7.34) contain the problem of stability. The integral of energy becomes, on substitution from (7.53) and (7.531),

(7.55) $$v\dot{\alpha} - \dot{v}\alpha = 2v\beta\omega + \delta h,$$

exactly the same result as in § 7.4. If we eliminate $(v\dot{\alpha} - \dot{v}\alpha)$ between (7.543) and (7.55), we obtain

(7.551) $$\ddot{\beta} + \beta\,[v^2K_{(1,1)} + V_{mn}\nu^m\nu^n + 3\omega^2 - \Omega^2] - 2\dot{\gamma}\Omega + \gamma\,[v^2K_{(1,2)} + V_{mn}\nu^m\rho^n - \dot{\Omega}] + 2\kappa\delta h = 0.$$

This equation and (7.544) now contain the problem, α being given by (7.55) and therefore expressible as in (7.454). Now from (7.53)

(7.56) $$\eta^2 = \alpha^2 + \beta^2 + \gamma^2,$$

and thus for stability it is necessary and sufficient that α, β and γ should all remain permanently small. We may therefore state the result :

THEOREM XXX (K).—*In order that the motion of a holonomic conservative system with three degress of freedom may be stable in the kinematical sense, it is necessary and sufficient that the values of* β *and* γ *as solutions of the differential equations* (7.544) *and* (7.551) *and the value of* α *as given by* (7.454) *should all be permanently small, for arbitrary infinitesimal values of the constants* δh *and* A.

Let us consider the case of steady motion. All the co-efficients in (7.544) and (7.551) are then constants and $\dot{\Omega}$ is zero. These equations may be written

(7.57) $$\begin{cases} \ddot{\beta} + p_1\beta - p_2\dot{\gamma} + p_3\gamma + 2\kappa\delta h = 0, \\ \ddot{\gamma} + p_4\gamma + p_2\dot{\beta} + p_3\beta = 0, \end{cases}$$

where p_1, p_2, p_3 and p_4 are constants defined by

(7.571)
$$\begin{cases} p_1 = v^2 K_{(1,1)} + V_{mn}\nu^m\nu^n + 3\omega^2 - \Omega^2, \\ p_2 = 2\Omega, \\ p_3 = v^2 K_{(1,2)} + V_{mn}\nu^m\rho^n, \\ p_4 = v^2 K_{(2,2)} + V_{mn}\rho^m\rho^n - \Omega^2. \end{cases}$$

If we substitute

(7.572)
$$\beta = Be^{nt} + B_1, \quad \gamma = Ce^{nt} + C_1,$$

we have for n the equation

(7.573)
$$\begin{vmatrix} n^2 + p_1, & -np_2 + p_3 \\ np_2 + p_3, & n^2 + p_4 \end{vmatrix} = 0,$$

or

(7.574)
$$n^4 + n^2(p_1 + p_4 + p_2^2) + (p_1 p_4 - p_3^2) = 0.$$

Thus for stable values of β and γ as solutions of (7.57) it is necessary and sufficient that

(7.575)
$$\begin{cases} p_1 + p_4 + p_2^2 > 0, \\ (p_1 + p_4 + p_2^2)^2 > 4(p_1 p_4 - p_3^2) > 0, \end{cases}$$

these being the conditions that the values of n^2 should be real, negative and distinct. For B_1 we have the equations

(7.576)
$$\begin{cases} p_1 B_1 + p_3 C_1 + 2\kappa\delta h = 0, \\ p_3 B_1 + p_4 C_1 = 0, \end{cases}$$

so that

(7.577)
$$B_1 = -\frac{2p_4\kappa\delta h}{p_1 p_4 - p_3^2}.$$

Now, assuming that (7.575) are satisfied, we easily see that the value of α given by (7.454) is permanently small for a non-zero value of δh if and only if

(7.578)
$$p_1 p_4 - p_3^2 = 4\omega^2 p_4.$$

We may state our result :

THEOREM XXXI (K).—*In order that a steady motion of a holonomic conservative system with three degrees of freedom may be stable in the kinematical sense, it is necessary and sufficient that*

$$p_1 + p_4 + p_2^2 > 0,$$
$$(p_1 + p_4 + p_2^2)^2 > 4(p_1 p_4 - p_3^2) > 0,$$
$$p_1 p_4 - p_3^2 = 4\omega^2 p_4,$$

where p_1, p_2, p_3 and p_4 are defined by (7.571).†

This result may be applied directly to the case of a particle in three dimensional Euclidean space having a steady motion in a circle. Here $K_{(1,1)}$, $K_{(1,2)}$, $K_{(2,2)}$ and Ω are all

† For brevity we have considered the general case where p_2 and p_3 are not both zero. If $p_2 = p_3 = 0$, necessary and sufficient conditions are $p_1 > 0$, $p_4 > 0$, $p_1 = 4\omega^2$.

zero and the necessary and sufficient conditions for stability in the kinematical sense may be written

$$(7.58)\quad \begin{cases} V_{mn}\nu^m\nu^n + V_{mn}\rho^m\rho^n + 3\omega^2 > 0, \\ (V_{mn}\nu^m\nu^n + V_{mn}\rho^m\rho^n + 3\omega^2)^2 > 4[(V_{mn}\nu^m\nu^n + 3\omega^2)V_{mn}\rho^m\rho^n - (V_{mn}\nu^m\rho^n)^2] > 0, \\ V_{mn}\nu^m\nu^n \,.\, V_{st}\rho^s\rho^t - (V_{mn}\nu^m\rho^n)^2 = \omega^2 V_{mn}\rho^m\rho^n. \end{cases}$$

If we use cylindrical co-ordinates (r, z, ϕ), r being the distance from a line through the centre of the circle perpendicular to its plane and z the distance from its plane, these conditions become

$$(7.581)\quad \begin{cases} V_{rr} + V_{zz} + 3\omega^2 > 0, \\ (V_{rr} + V_{zz} + 3\omega^2)^2 > 4[(V_{rr} + 3\omega^2)V_{zz} - V_{rz}^2] > 0, \\ V_{rr}V_{zz} - V_{rz}^2 = \omega^2 V_{zz}, \end{cases}$$

where ω is the angular velocity of the particle in its circular motion and subscripts denote partial derivatives.† This simple example is easy to discuss directly and the conditions in the above form are immediately verified. We shall have more to say of this example in § 8.4.

§ 7.6. *The general method of resolution along the normals.*

We shall now proceed to the general case of N degrees of freedom, without assuming for the present that the system is necessarily conservative. The method is essentially a generalisation of the method of moving axes, based on the Frenet-Serret formulæ of § 2.7. These formulæ may be exhibited in the compact form

$$(7.61)\quad \overline{\lambda}^r_{(M)} = \kappa_{(M+1)}\lambda^r_{(M+1)} - \kappa_{(M)}\lambda^r_{(M-1)},$$

where M is zero or any positive integer and where $\kappa_{(P)}$ vanishes identically unless P is one of the numbers $1, 2, \ldots, N-1$. We can also write these equations in the form

$$(7.611)\quad \hat{\lambda}^r_{(M)} = \omega_{(M+1)}\lambda^r_{(M+1)} - \omega_{(M)}\lambda^r_{(M-1)}, \qquad (M = 0, 1, \ldots),$$

where $\omega_{(P)}$ vanishes identically unless P is one of the numbers $1, 2, \ldots, N-1$. The quantities $\omega_{(1)}, \omega_{(2)}, \ldots \omega_{(N-1)}$ may be called the *angular velocities* of the undisturbed motion, each being equal to the corresponding curvature multiplied by the velocity.

Taking the contravariant time-flux of (7.611) and then substituting from the same equations, we obtain

$$(7.612)\quad \hat{\hat{\lambda}}^r_{(M)} = \omega_{(M+1)}\omega_{(M+2)}\lambda^r_{(M+2)} + \dot{\omega}_{(M+1)}\lambda^r_{(M+1)} - [(\omega_{(M+1)})^2 + (\omega_{(M)})^2]\lambda^r_{(M)} - \dot{\omega}_{(M)}\lambda^r_{(M-1)} + \omega_{(M)}\omega_{(M-1)}\lambda^r_{(M-2)}, \qquad (M = 0, 1, \ldots).$$

Now let us write the disturbance vector in the form

$$(7.62)\quad \eta^r = \alpha_{(0)}\lambda^r_{(0)} + \alpha_{(1)}\lambda^r_{(1)} + \ldots + \alpha_{(N-1)}\lambda^r_{(N-1)},$$

so that $\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(N-1)}$ are the components of the disturbance vector in the

† We have assumed that V_{rz} is not zero.

directions of the tangent, first normal, second normal, etc., respectively. From these equations we obtain

(7.621) $$\hat{\eta}^r = \sum_{M=0}^{N-1} (\dot{\alpha}_{(M)} \lambda^r_{(M)} + \alpha_{(M)} \hat{\lambda}^r_{(M)}),$$

and

(7.622) $$\hat{\hat{\eta}}^r = \sum_{M=0}^{N-1} (\ddot{\alpha}_{(M)} \lambda^r_{(M)} + 2\dot{\alpha}_{(M)} \hat{\lambda}^r_{(M)} + \alpha_{(M)} \hat{\hat{\lambda}}^r_{(M)}).$$

Thus

(7.63) $$a_{rs}\hat{\hat{\eta}}^r \lambda^s_{(P)} = \sum_{M=0}^{N-1} \ddot{\alpha}_{(M)} a_{rs}\lambda^r_{(M)} \lambda^s_{(P)} + 2 \sum_{M=0}^{N-1} \dot{\alpha}_{(M)} a_{rs}\hat{\lambda}^r_{(M)} \lambda^s_{(P)} + \sum_{M=0}^{N-1} \alpha_{(M)} a_{rs} \hat{\hat{\lambda}}^r_{(M)} \lambda^s_{(P)},$$

$$(P = 0, 1, \ldots, N-1).$$

Hence, using the fact that

(7.631) $$a_{rs}\lambda^r_{(M)} \lambda^s_{(P)} = \begin{cases} 1 \text{ if } P = M, \\ 0 \text{ if } P \neq M, \end{cases}$$

we have the kinematical equations

(7.632) $$a_{rs}\hat{\hat{\eta}}^r\lambda^s_{(P)} = \ddot{\alpha}_{(P)} + 2\,[\dot{\alpha}_{(P-1)}\,\omega_{(P)} - \dot{\alpha}_{(P+1)}\,\omega_{(P+1)}] + \alpha_{(P-2)}\,\omega_{(P-1)}\,\omega_{(P)} + \alpha_{(P-1)}\,\dot{\omega}_{(P)} - \alpha_{(P)}\,[(\omega_{(P+1)})^2 + (\omega_{(P)})^2] - \alpha_{(P+1)}\,\dot{\omega}_{(P+1)} + \alpha_{(P+2)}\,\omega_{(P+2)}\,\omega_{(P+1)},$$

$$(P = 0, 1, \ldots, N-1).$$

Now from (7.15) we have the dynamical equations

(7.64) $$a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(P)} = -\,v^2 G_{rmsn}\lambda^r_{(P)}\,\lambda^m_{(0)}\,\eta^s\lambda^n_{(0)} + Q_{rs}\lambda^r_{(P)}\,\eta^s,$$

$$(P = 0, 1, \ldots, N-1),$$

or, with the notation of § 6.6,

(7.641) $$a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(P)} = -\,v^2 \sum_{M=0}^{N-1} \alpha_{(M)}\,K_{(P,\,M)} + \sum_{M=0}^{N-1} \alpha_{(M)}\,W_{(P,\,M)},$$

$$(P = 0, 1, \ldots, N-1).$$

But by (3.123)

(7.65) $$Q^r = \dot{v}\lambda^r_{(0)} + v\,\omega_{(1)}\,\lambda^r_{(1)},$$

and therefore

(7.651) $$\begin{cases} Q_r\lambda^r_{(0)} = \dot{v}, \\ Q_r\lambda^r_{(1)} = v\,\omega_{(1)}, \\ Q_r\lambda^r_{(P)} = 0. \qquad (P = 2, 3, \ldots, N-1). \end{cases}$$

If we differentiate these equations with respect to t and use (7.611), we obtain

(7.652) $$\begin{cases} vW_{(0,\,0)} = \ddot{v} - v\,(\omega_{(1)})^2, \\ vW_{(1,\,0)} = 2\dot{v}\,\omega_{(1)} + v\dot{\omega}_{(1)}, \\ W_{(2,\,0)} = \omega_{(1)}\,\omega_{(2)}, \\ W_{(P,\,0)} = 0, \qquad (P = 3, 4, \ldots, N-1). \end{cases}$$

When we substitute from these equations in (7.641) and remember that $K_{(0,P)}$ and $K_{(P,0)}$ both vanish, we find

$$(7.653)\quad \left\{\begin{aligned} a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(0)} &= \frac{\alpha_{(0)}}{v}[\ddot{v} - v(\omega_{(1)})^2] + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(0,M)},\\ a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(1)} &= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(1,M)} + \frac{\alpha_{(0)}}{v}[2\dot{v}\omega_{(1)} + v\dot{\omega}_{(1)}] + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(1,M)},\\ a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(2)} &= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(2,M)} + \alpha_{(0)}\omega_{(1)}\omega_{(2)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(2,M)},\\ a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(P)} &= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(P,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(P,M)},\\ &\qquad\qquad (P = 3, 4, \ldots, N-1). \end{aligned}\right.$$

If we compare these equations with (7.632), we obtain the equations for the components of the disturbance vector along the normals in the following form :

$$(7.654)\quad \ddot{\alpha}_{(0)} - 2\dot{\alpha}_{(1)}\omega_{(1)} - \alpha_{(1)}\dot{\omega}_{(1)} + \alpha_{(2)}\omega_{(1)}\omega_{(2)} = \frac{\alpha_{(0)}\ddot{v}}{v} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(0,M)},$$

$$(7.655)\quad \ddot{\alpha}_{(1)} + 2\dot{\alpha}_{(0)}\omega_{(1)} - 2\dot{\alpha}_{(2)}\omega_{(2)} - \alpha_{(1)}[(\omega_{(1)})^2 + (\omega_{(2)})^2] - \alpha_{(2)}\dot{\omega}_{(2)} + \alpha_{(3)}\omega_{(2)}\omega_{(3)}$$
$$= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(1,M)} + 2\frac{\dot{v}}{v}\alpha_{(0)}\omega_{(1)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(1,M)},$$

$$(7.656)\quad \ddot{\alpha}_{(2)} + 2\dot{\alpha}_{(1)}\omega_{(2)} - 2\dot{\alpha}_{(3)}\omega_{(3)} + \alpha_{(1)}\dot{\omega}_{(2)} - \alpha_{(2)}[(\omega_{(2)})^2 + (\omega_{(3)})^2] - \alpha_{(3)}\dot{\omega}_{(3)} + \alpha_{(4)}\omega_{(3)}\omega_{(4)}$$
$$= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(2,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(2,M)},$$

$$(7.657)\quad \ddot{\alpha}_{(P)} + 2\dot{\alpha}_{(P-1)}\omega_{(P)} - 2\dot{\alpha}_{(P+1)}\omega_{(P+1)} + \alpha_{(P-2)}\omega_{(P-1)}\omega_{(P)}$$
$$+ \alpha_{(P-1)}\dot{\omega}_{(P)} - \alpha_{(P)}[(\omega_{(P)})^2 + (\omega_{(P+1)})^2] - \alpha_{(P+1)}\dot{\omega}_{(P+1)} + \alpha_{(P+2)}\omega_{(P+1)}\omega_{(P+2)}$$
$$= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(P,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(P,M)},$$
$$(P = 3, 4, \ldots, N-1).$$

The importance of these equations lies in the fact that in the case of steady motion all the coefficients are constant.†

Since

$$(7.658)\quad \eta^2 = (\alpha_{(0)})^2 + (\alpha_{(1)})^2 + \ldots + (\alpha_{(N-1)})^2,$$

we may state the following result :

THEOREM XXXII (K).—*In order that the motion of a holonomic system may be stable in the kinematical sense, it is necessary and sufficient that the values of* $\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(N-1)}$ *as solutions of the differential equations* (7.654) *to* (7.657) *should be permanently small. In the case of steady motion the coefficients in these equations are constants.*

† This is what ROUTH demanded of a steady motion ; *cf. Stability of Motion*, 2.

We have not so far assumed the force system to be conservative. This we shall now assume, but shall not restrict ourselves to steady motions. For a conservative system equation (7.654) may be replaced by the integral of energy (7.34), which is easily seen to reduce to

(7.66)
$$v\dot{\alpha}_{(0)} - \dot{v}\alpha_{(0)} = 2v\alpha_{(1)}\omega_{(1)} + \delta h,$$

which gives

(7.661)
$$\alpha_{(0)} = 2v\int_o^t \alpha_{(1)}\kappa_{(1)}dt + v\delta h\int_o^t \frac{dt}{v^2} + Av,$$

where A is an infinitesimal constant equal to the initial value of $\alpha_{(0)}/v$. By means of (7.66) we can simplify (7.655) to the form

(7.662)
$$\ddot{\alpha}_{(1)} - 2\dot{\alpha}_{(2)}\omega_{(2)} + \alpha_{(1)}\left[3(\omega_{(1)})^2 - (\omega_{(2)})^2\right] - \alpha_{(2)}\dot{\omega}_{(2)} + \alpha_{(3)}\omega_{(2)}\omega_{(3)} + 2\kappa_{(1)}\delta h$$
$$= -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(1,\,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(1,\,M)}.$$

Thus we may state the result :

THEOREM XXXIII (K).—*In order that the motion of a holonomic conservative system may be stable in the kinematical sense, it is necessary and sufficient that the values of* $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-1)}$ *as solutions of the differential equations* (7.662), (7.656) *and* (7.657) *and the value of* $\alpha_{(0)}$ *as given by* (7.661) *should be permanently small, for all arbitrary infinitesimal values of the constants* δh *and* A.

If the curve of the undisturbed motion is a geodesic of the manifold of configurations (which is the case when the forces are zero), $\kappa_{(1)}$ vanishes. Then the motion is unstable if, for some pair of values of δh and A, the value of $\alpha_{(0)}$ given by

(7.67)
$$\alpha_{(0)} = v\delta h\int_o^t \frac{dt}{v^2} + Av$$

is not bounded. If v does not vanish for any finite value of t, it is not difficult to show that the first part of this expression is not bounded.† For suppose that it is bounded, so that

(7.671)
$$v\int_0^t \frac{dt}{v^2} < M,$$

v being without loss of generality supposed to be positive. If we write

(7.672)
$$F(t) = \int_0^t \frac{dt}{v^2},$$

we have

(7.673)
$$F'(t) = \frac{1}{v^2},$$

† The following method is due to Prof. C. H. ROWE, F.T.C.D.

or

(7.674) $$v = [F'(t)]^{-\frac{1}{2}},$$

so that

(7.675) $$v \int_0^t \frac{dt}{v^2} = F(t)\,[F'(t)]^{-\frac{1}{2}} = \left[-\frac{d}{dt}\frac{1}{F(t)}\right]^{-\frac{1}{2}}.$$

Thus, by (7.671),

(7.676) $$-\frac{d}{dt}\frac{1}{F(t)} > M^{-2}.$$

Hence $1/F(t)$ must vanish for a finite value of t, which is impossible. Therefore (7.671) is not true. Thus the value of $\alpha_{(0)}$ is not bounded and there is instability. Hence we have the result :—

THEOREM XXXIV (K) :—*Every natural motion of a holonomic conservative system which is a natural motion under no forces (or, more generally, which takes place along a geodesic of the manifold of configurations) is unstable in the kinematical sense, provided that the velocity does not vanish for any finite value of t.*

The motion of a sleeping top is a natural motion under no forces. It is therefore unstable in the kinematical sense, a fact intuitively obvious when we contemplate an increase of spin without moving the axis.

Another interesting case is that of steady motion in which all the curvatures except the first vanish. Considering only a conservative system, we have from (7.661)

(7.68) $$\alpha_{(0)} = 2v\kappa_{(1)} \int_0^t \alpha_{(1)}\,dt + \frac{t\delta h}{v} + Av,$$

and from (7.662), (7.656) and (7.657)

(7.681) $$\ddot{\alpha}_{(1)} + 3\alpha_{(1)}(\omega_{(1)})^2 + 2\kappa_{(1)}\delta h + v^2 \sum_{M=1}^{N-1} \alpha_{(M)} K_{(1, M)} - \sum_{M=1}^{N-1} \alpha_{(M)} W_{(1, M)} = 0,$$

(7.682) $$\ddot{\alpha}_{(P)} + v^2 \sum_{M=1}^{N-1} \alpha_{(M)} K_{(P, M)} - \sum_{M=1}^{N-1} \alpha_{(M)} W_{(P, M)} = 0, \quad (P = 2, 3, \ldots, N-1).$$

These equations will have solutions of the type

(7.683) $$\alpha_{(\rho)} = B_{(\rho, 1)} e^{p_{(1)}t} + B_{(\rho, 2)} e^{p_{(2)}t} + \ldots + B_{(\rho, N-1)} e^{p_{(N-1)}t} + C_{(\rho)},$$

where $B_{(\rho, \sigma)}$, $p_{(\rho)}$, $C_{(\rho)}$ are constants, $p_{(\rho)}$ being the roots of the determinantal equation

(7.684)
$$\begin{vmatrix} p^2 + v^2K_{(1,1)} - W_{(1,1)} + 3(\omega_{(1)})^2, & v^2K_{(1,2)} - W_{(1,2)}, & \ldots, & v^2K_{(1,N-1)} - W_{(1,N-1)} \\ v^2K_{(2,1)} - W_{(2,1)}, & p^2 + v^2K_{(2,2)} - W_{(2,2)}, & \ldots, & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot \\ v^2K_{(N-1,1)} - W_{(N-1,1)}, & \ldots, & & p^2 + v^2K_{(N-1,N-1)} - W_{(N-1,N-1)} \end{vmatrix} = 0,$$

while $C_{(\rho)}$ satisfy the equations

$$(7.685)\quad \begin{cases} C_{(1)}[v^2K_{(1,1)} - W_{(1,1)} + 3(\omega_{(1)})^2] + 2\kappa_{(1)}\delta h + C_{(2)}[v^2K_{(1,2)} - W_{(1,2)}] + \ldots \\ \qquad\qquad + C_{(N-1)}[v^2K_{(1,N-1)} - W_{(1,N-1)}] = 0, \\ C_{(1)}[v^2K_{(2,1)} - W_{(2,1)}] + C_{(2)}[v^2K_{(2,2)} - W_{(2,2)}] + \ldots \\ \qquad\qquad + C_{(N-1)}[v^2K_{(2,N-1)} - W_{(2,N-1)}] = 0, \\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ C_{(1)}[v^2K_{(N-1,1)} - W_{(N-1,1)}] + C_{(2)}[v^2K_{(N-1,2)} - W_{(N-1,2)}] + \ldots \\ \qquad\qquad + C_{(N-1)}[v^2K_{(N-1,N-1)} - W_{(N-1,N-1)}] = 0. \end{cases}$$

To obtain permanently small values for $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-1)}$ it is necessary that the roots of (7.684) should be purely imaginary. If $\alpha_{(0)}$, given by (7.68), is to be permanently small, it is evident that we must have

$$(7.686)\qquad 2v\kappa_{(1)}C_{(1)} + \frac{\delta h}{v} = 0,$$

or

$$(7.687)\qquad C_{(1)} = -\frac{\delta h}{2v^2\kappa_{(1)}}.$$

Thus

$$(7.688)\qquad 3C_{(1)}(\omega_{(1)})^2 + 2\kappa_{(1)}\delta h = \tfrac{1}{2}\kappa_{(1)}\delta h = -C_{(1)}(\omega_{(1)})^2,$$

and thus from (7.685) we derive

$$(7.689)\quad \begin{vmatrix} v^2K_{(1,1)} - W_{(1,1)} - (\omega_{(1)})^2, & v^2K_{(1,2)} - W_{(1,2)}, & \ldots, & v^2K_{(1,N-1)} - W_{(1,N-1)} \\ v^2K_{(2,1)} - W_{(2,1)}, & v^2K_{(2,2)} - W_{(2,2)}, & \ldots, & v^2K_{(2,N-1)} - W_{(2,N-1)} \\ \cdot & \cdot & \cdot & \cdot \\ v^2K_{(N-1,1)} - W_{(N-1,1)} & & \ldots, & v^2K_{(N-1,N-1)} - W_{(N-1,N-1)} \end{vmatrix} = 0.$$

Since the vanishing of the second curvature implies the vanishing of all curvatures of higher order, we may state the result :—

THEOREM XXXV (K) :—*In order that a steady motion with vanishing second curvature of a holonomic conservative system may be stable in the kinematical sense, it is necessary and sufficient that the roots of the determinantal equation* (7.684) *should all be purely imaginary and that* (7.689) *should be true.*

A particular case of special interest, where the variables are separated, may also be considered. We shall adopt a semi-geometrical point of view.

In the classical discussion of vibrations about equilibrium, the infinitesimal quadric,

$$(7.69)\qquad V_{mn}\,dq^m\,dq^n = \textit{constant},$$

plays a fundamental part. In fact, it is necessary and sufficient for stability that this quadric should be of the closed (or ellipsoidal) type, the independent normal vibrations taking place along its principal axes. In the case of vibrations about a state of motion there is a corresponding quadric of importance to which we shall now proceed.

Being given a trajectory, let A be any point on it with co-ordinates q^r and let B be any neighbouring point in the manifold of configurations with co-ordinates $(q^r + \xi^r)$. Consider the locus of the point B (A being fixed) if

$$v^2 G_{mnst} \xi^m \lambda^n_{(0)} \xi^s \lambda^t_{(0)} + V_{ms} \xi^m \xi^s = \textit{constant}. \tag{7.691}$$

This is an infinitesimal quadric surface; we shall call it *the stability quadric*. The *plane conjugate to the direction* $\xi^r_{(1)}$ will have the equation

$$v^2 G_{mnst} \xi^m \lambda^n_{(0)} \xi^s_{(1)} \lambda^t_{(0)} + V_{ms} \xi^m \xi^s_{(1)} = 0, \tag{7.692}$$

and the directions $\xi^r_{(1)}$ and $\xi^r_{(2)}$ will be *conjugate* if

$$v^2 G_{mnst} \xi^m_{(1)} \lambda^n_{(0)} \xi^s_{(2)} \lambda^t_{(0)} + V_{ms} \xi^m_{(1)} \xi^s_{(2)} = 0. \tag{7.693}$$

A *principal direction* of the stability quadric is a direction which is perpendicular to its conjugate plane (*i.e.*, perpendicular to all the directions which are conjugate to it), the analytical conditions that ξ^r should define a principal direction being

$$v^2 G_{mnst} \xi^m \lambda^n_{(0)} \lambda^t_{(0)} + V_{ms} \xi^m = \theta a_{ms} \xi^m, \tag{7.694}$$

θ being a root of the determinantal equation

$$\| v^2 G_{mnst} \lambda^n_{(0)} \lambda^t_{(0)} + V_{ms} - \theta a_{ms} \| = 0. \tag{7.695}$$

It is easily seen that any two principal directions are perpendicular to one another.

In the case of a steady motion for which the second curvature vanishes (and to this case we confine ourselves), we see from (6.621) and (7.652) that the tangential direction is conjugate to every normal direction. Hence the tangent to the trajectory is a principal direction. *Let us hypothesise that the first normal is also a principal direction.* The remaining principal directions will lie in the normal plane and will be perpendicular to the first normal. Now, on account of the vanishing of the second curvature, we are at liberty to choose $\lambda^r_{(2)}, \lambda^r_{(3)}, \ldots, \lambda^r_{(N-1)}$ arbitrarily except for the restrictions laid down in § 2.7. At some one point of the trajectory let us choose these vectors coincident with the remaining principal directions. Then, at that point,

$$v^2 K_{(P,Q)} - W_{(P,Q)} = 0, \qquad (P \neq Q;\ P, Q = 0, 1, \ldots, N-1), \tag{7.696}$$

and, since the motion is steady, these relations will hold true at every point of the trajectory. These are the conditions that $\lambda^r_{(0)}, \lambda^r_{(1)}, \ldots, \lambda^r_{(N-1)}$ should be principal directions of the stability quadric.

Equations (7.681) and (7.682) reduce to

$$\left\{\begin{array}{l}\ddot{\alpha}_{(1)} + \alpha_{(1)}[v^2 K_{(1,1)} - W_{(1,1)} + 3(\omega_{(1)})^2] + 2\kappa_{(1)}\,\delta h = 0,\\ \ddot{\alpha}_{(P)} + \alpha_{(P)}[v^2 K_{(P,P)} - W_{(P,P)}] = 0. \qquad (P = 2, 3, \ldots, N-1),\end{array}\right. \tag{7.697}$$

in which the variables are separated. We deduce at once the generalisation of Theorem XXIX (§ 7.4):

THEOREM XXXVI (K):—*A steady motion of a holonomic conservative system, for which the second curvature of the trajectory vanishes and the first normal is a principal direction of the stability quadric, is stable in the kinematical sense if and only if*

$$(7.698)\qquad \begin{cases} v^2 K_{(1,1)} + V_{mn}\lambda^m_{(1)}\lambda^n_{(1)} - v^2(\kappa_{(1)})^2 = 0, \\ v^2 K_{(P,P)} + V_{mn}\lambda^m_{(P)}\lambda^n_{(P)} > 0, \qquad (P = 2, 3, \ldots, N-1), \end{cases}$$

$K_{(P,P)}$ $(P = 1, 2, \ldots, N-1)$ *being the Riemannian curvatures of the manifold of configurations corresponding to the two-space elements defined by the tangent to the trajectory and the other principal directions of the stability quadric,* $\lambda^r_{(1)}$ *being the unit vector in the direction of the first normal and* $\lambda^r_{(2)}, \lambda^r_{(3)}, \ldots, \lambda^r_{(N-1)}$ *being unit vectors in the remaining principal directions of the stability quadric.*

The periods of the stable vibrations are

$$(7.699)\qquad \begin{cases} \pi/v\kappa_{(1)}, \\ 2\pi/(v^2 K_{(P,P)} + V_{mn}\lambda^m_{(P)}\lambda^n_{(P)})^{\frac{1}{2}}, \qquad (P = 2, 3, \ldots, N-1). \end{cases}$$

CHAPTER VIII.

STABILITY IN THE KINEMATICO-STATICAL SENSE.

§ 8.1. *Equations for the components of the disturbance vector.*

Stability in the kinematico-statical sense is probably the most interesting from a physical point of view, because we are chiefly interested in the question as to whether on account of the disturbance the system passes through configurations widely different from those through which it would have passed without being disturbed. This type of stability is correspondingly the most difficult to discuss, and we shall confine ourselves throughout to conservative systems.

Let η^r be the infinitesimal vector drawn from the point P of the undisturbed trajectory C to the corresponding point P^* of the disturbed trajectory C^*, the correspondence between P and P^* being fixed by the condition of perpendicularity

$$(8.11)\qquad a_{mn}\dot{q}^m\eta^n = 0.$$

Let O and O^* be fixed corresponding points on C and C^* respectively. There are associated with any point P of C four quantities

s, the length of the arc OP;

s^*, the length of the arc O^*P^*;

t, the time corresponding to P for the natural motion along C;

t^*, the time corresponding to P^* for the natural motion along C^*.

Between these quantities we shall now establish relations sufficient to express all in terms of one when the natural motion along C is known. The operational symbols for derivatives, time-fluxes, etc., are used only for the variables s and t. When s^* or t^* is the independent variable, the expression is written explicitly.

Since PP^* is normal to C, we have†

(8.12) $$s^* - s = -\int_O^P a_{rs}\kappa^r\eta^s\,ds,$$

where κ^r is the contravariant vector of first curvature. Hence

(8.121) $$\frac{ds^*}{ds} = 1 - a_{rs}\kappa^r\eta^s.$$

But by (3.123), since $f^r = -a^{rs}V_s$, and by virtue of (8.11), we find

(8.122) $$v^2 a_{rs}\kappa^r\eta^s = -V_m\eta^m;$$

so that we have

(8.123) $$\frac{ds^*}{ds} = 1 + V_m\eta^m/v^2.$$

Let h denote the total energy for C and $(h + \delta h)$ that for C^*, δh being infinitesimal. Then

(8.13) $$T^* + V^* - T - V = \delta h,$$

so that, if we write,

(8.131) $$T^* = T(1 + 2\tau),$$

we have, since $v^2 = 2T$,

(8.132) $$v^2\tau + V_m\eta^m = \delta h.$$

Squares and higher powers of τ may therefore be neglected. Thus

(8.133) $$\frac{dt^*}{dt} = \left(\frac{T}{T^*}\right)^{1/2}\frac{ds^*}{ds} = (1-\tau)(1 + V_m\eta^m/v^2)$$
$$= 1 + \frac{2V_m\eta^m}{v^2} - \frac{\delta h}{v^2}.$$

Now the dynamical equations of C^* are

(8.14) $$\frac{d^2q^{*r}}{dt^{*2}} + \left\{\begin{matrix} m\ n \\ r \end{matrix}\right\}^* \frac{dq^{*m}}{dt^*}\frac{dq^{*n}}{dt^*} = (Q^r)^*,$$

or, by change of the independent variable to t,

(8.141) $$\left(\frac{dt}{dt^*}\right)^2\left[\ddot{q}^{*r} + \left\{\begin{matrix} m\ n \\ r \end{matrix}\right\}^* \dot{q}^{*m}\dot{q}^{*n}\right] - \left(\frac{dt}{dt^*}\right)^3\frac{d^2t^*}{dt^2}\,\dot{q}^{*r} = (Q^r)^*,$$

† *Cf.* "The First and Second Variations of the Length-Integral in Riemannian Space," *Proc. Lond. Math. Soc.* (2), 25 (1926), 247. The notation is suitably altered.

from which, by the help of (8.133) and with neglect of second order quantities, we derive

$$\ddot{q}^{*r} + \left\{\begin{matrix} m\ n \\ r \end{matrix}\right\}^* \dot{q}^{*m}\dot{q}^{*n} - (Q^r)^* - \frac{2}{v^2}(2V_m\eta^m - \delta h)f^r - \dot{q}^r \frac{d}{dt}\left(\frac{2V_m\eta^m - \delta h}{v^2}\right) = 0. \tag{8.142}$$

But in § 7.1 we occupied ourselves with the analytical problem of estimating the value of

$$\ddot{q}^{*r} + \left\{\begin{matrix} m\ n \\ r \end{matrix}\right\}^* \dot{q}^{*m}\dot{q}^{*n} - (Q^r)^*, \tag{8.143}$$

where $q^{*r} = q^r + \eta^r$, and found that it was equal to

$$\hat{\hat{\eta}}^r + G^r_{msn}\eta^s\dot{q}^m\dot{q}^n - Q^r_s\eta^s. \tag{8.144}$$

Substituting in (8.142) and putting in V instead of Q (with change of sign), we have, on changing to the covariant form,

$$a_{rs}\hat{\hat{\eta}}^s + G_{rmsn}\dot{q}^m\eta^s\dot{q}^n + V_{rs}\eta^s + \frac{2}{v^2}(2V_m\eta^m - \delta h)V_r - a_{rs}\dot{q}^s\frac{d}{dt}\left(\frac{2V_m\eta^m - \delta h}{v^2}\right) = 0. \tag{8.15}$$

This equation may be called *the tensorial equation for the components of the disturbance vector in the kinematico-statical sense.*

§ 8.2. *Equation for the magnitude of the disturbance vector.*

If, as in § 7.2, we write μ^r for the unit vector co-directional with the disturbance vector η^r, then (7.24) is true. If then we multiply (8.15) by μ^r and sum as indicated, we have

$$\ddot{\eta} - \eta a_{mn}\hat{\mu}^m\hat{\mu}^n + V_{rs}\mu^r\eta^s + G_{rmsn}\mu^r\dot{q}^m\eta^s\dot{q}^n + \frac{2}{v^2}(2V_m\eta^m - \delta h)V_r\mu^r = 0, \tag{8.21}$$

the last term having disappeared by virtue of (8.11). This may be written

$$\ddot{\eta} + \eta\left(G_{mnst}\,\mu^m\dot{q}^n\mu^s\dot{q}^t - \hat{\mu}^2 + V_{mn}\mu^m\mu^n + \frac{4}{v^2}[V_m\mu^m]^2\right) = \frac{2\delta h}{v^2}V_m\mu^m. \tag{8.22}$$

This is *the invariant equation for the magnitude of the disturbance in the kinematico-statical sense.*

It is interesting to compare this equation with (7.26) and note that there is an additional term $4(V_m\mu^m)^2/v^2$ which is positive and therefore contributes to stability. This was to be expected, since stability in the kinematical sense implies stability in the kinematico-statical sense, but not conversely. The presence of this term and that on the right-hand side prevents us from stating a theorem analogous to Theorem XXVI.

§ 8.3. *The case of a conservative system with two degrees of freedom.*

This case is very simple because the disturbance vector is along the normal to the curve, so that

$$\mu^r = \varepsilon\nu^r, \qquad \varepsilon = \pm 1. \tag{8.31}$$

We have then

(8.32) $$\hat{\mu}^r = \varepsilon \hat{\nu}^r,$$

from which, by (7.41), we find

(8.33) $$\hat{\mu} = \hat{\nu} = \omega = v\kappa.$$

Also, equating normal components of force and acceleration, we have

(8.34) $$V_m \mu^m = \varepsilon V_m \nu^m = -\varepsilon v^2 \kappa = -\varepsilon v \omega,$$

and thus (8.22) becomes

(8.35) $$\ddot{\eta} + \eta\,(v^2 K + V_{mn} \nu^m \nu^n + 3v^2\kappa^2) = -2\varepsilon\kappa\delta h,$$

where K is the Gaussian curvature of the manifold of configurations.

Now η is by definition a positive quantity. Let us put $\beta = a_{mn}\eta^m\nu^n$, so that $\beta = \eta$ when ε is positive and $\beta = -\eta$ when ε is negative. The question of stability is then a question of the permanent smallness of β where

(8.36) $$\ddot{\beta} + \beta\,(v^2 K + V_{mn}\,\nu^m \nu^n + 3v^2\kappa^2) + 2\kappa\delta h = 0,$$

which is the same as the equation (7.453), the coefficient of β being the "coefficient of stability." We may state the result :

THEOREM XXXVII (K) :—*In order that the motion of a holonomic conservative system with two degrees of freedom may be stable in the kinematico-statical sense, it is necessary and sufficient that the solution of the differential equation* (8.36) *should be permanently small, for arbitrary values of the infinitesimal constant* δh.

When δh is zero, we may apply Sturm's theorem to (8.36) and obtain the result :†

THEOREM XXXVIII (K) :—*If all along a natural trajectory of a holonomic conservative system with two degrees of freedom the coefficient of stability is positive, then the motion is stable in the kinematico-statical sense for all disturbances which do not change the total energy.*

In the case of a steady motion, the coefficient of stability and the curvature (κ) are constant along the trajectory and we have the result :

THEOREM XXXIX (K) :—*In order that a steady motion of a holonomic conservative system with two degrees of freedom may be stable in the kinematico-statical sense, it is necessary and sufficient that the coefficient of stability should be positive.*

† *Cf.* the remarks on the coefficient of stability in § 7.4. For a particle orbit in a plane the coefficient of stability may be written $\partial^2 V/\partial u^2 + 3v^2/\rho^2$, where u denotes the distance from the undisturbed curve and ρ the radius of curvature.

§ 8.4. *The case of a conservative system with three degrees of freedom.*

To develop the theory of this case we may borrow largely from § 7.5. From condition (8.11) it follows that α in (7.53) is zero and we have as in (7.533) and (7.534)

(8.41) $$a_{rs}\ddot{\eta}^r\nu^s = \ddot{\beta} - \beta(\omega^2 + \Omega^2) - 2\dot{\gamma}\Omega - \gamma\dot{\Omega},$$

and

(8.411) $$a_{rs}\ddot{\eta}^r\rho^s = \ddot{\gamma} - \gamma\Omega^2 + 2\dot{\beta}\Omega + \beta\dot{\Omega},$$

while from (8.15) we find

(8.42) $$a_{rs}\ddot{\eta}^r\nu^s = -G_{rmsn}\nu^r\dot{q}^m\eta^s\dot{q}^n - V_{rs}\nu^r\eta^s - \frac{2}{v^2}(2V_m\eta^m - \delta h)\,V_r\nu^r,$$

and

(8.421) $$a_{rs}\ddot{\eta}^r\rho^s = -G_{rmsn}\rho^r\dot{q}^m\eta^s\dot{q}^n - V_{rs}\rho^r\eta^s,$$

remembering that $V_r\rho^r$ is zero. Comparing these four equations, we have

(8.43) $$\ddot{\beta} - \beta(\omega^2 + \Omega^2) - 2\dot{\gamma}\Omega - \gamma\dot{\Omega} = -v^2\beta K_{(1,1)} - v^2\gamma K_{(1,2)} - \beta V_{mn}\nu^m\nu^n - \gamma V_{mn}\nu^m\rho^n - 4\beta\omega^2 - 2\kappa\,\delta h,$$

and

(8.431) $$\ddot{\gamma} - \gamma\Omega^2 + 2\dot{\beta}\Omega + \beta\dot{\Omega} = -v^2\beta K_{(1,2)} - v^2\gamma K_{(2,2)} - \beta V_{mn}\nu^m\rho^n - \gamma V_{mn}\rho^m\rho^n,$$

so that we obtain for β and γ precisely equations (7.544) and (7.551), a fact which is not surprising yet hardly obvious, because in § 7.5 we were dealing with a different correspondence.

Analogues to Theorems XXX and XXXI are immediately available :

THEOREM XL (K) :—*In order that the motion of a holonomic conservative system with three degrees of freedom may be stable in the kinematico-statical sense, it is necessary and sufficient that the values of* β *and* γ *as solutions of the differential equations* (7.544) *and* (7.551) *should be permanently small, for arbitrary values of the infinitesimal constant* δh.

THEOREM XLI (K) :—*In order that a steady motion of a holonomic conservative system with three degrees of freedom may be stable in the kinematico-statical sense, it is necessary and sufficient that*

(8.44) $$\begin{cases} p_1 + p_4 + p_2^2 > 0, \\ (p_1 + p_4 + p_2^2)^2 > 4(p_1p_4 - p_3^2) > 0, \end{cases}$$

where p_1, p_2, p_3 *and* p_4 *are defined by* (7.571).†

† These are the conditions when p_2 and p_3 are not both zero. When $p_2 = p_3 = 0$, necessary and sufficient conditions are $p_1 > 0$, $p_4 > 0$.

Applying this latter theorem to the problem at the end of § 7.5, we see that necessary and sufficient conditions for stability in the kinematico-statical sense are

$$(8.45)\qquad \begin{cases} V_{rr} + V_{zz} + 3\omega^2 > 0, \\ (V_{rr} + V_{zz} + 3\omega^2)^2 > 4\left[(V_{rr} + 3\omega^2)\,V_{zz} - V_{rz}^{\;2}\right] > 0, \end{cases}$$

where the subscripts denote partial derivatives.† The conditions may also be written in the form

$$(8.451)\qquad V_{zz} > 0,\ (V_{rr} + 3\omega^2)V_{zz} - V_{rz}^{\;2} > 0.$$

§ 8.5. *The general method of resolution along the normals.*

The results of the two preceding sections would lead us to suspect that for the discussion of kinematico-statical stability we would have precisely the same equations for the components of the disturbance vector along the normals as we had in the discussion of kinematical stability. This is actually the case, and we shall proceed to demonstrate it analytically.

Let us write

$$(8.51)\qquad \eta^r = \alpha_{(1)}\lambda^r_{(1)} + \alpha_{(2)}\lambda^r_{(2)} + \ldots + \alpha_{(N-1)}\lambda^r_{(N-1)},$$

so that $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-1)}$ are the components of the disturbance vector along the normals. We note that this is the same equation as (7.62) when $\alpha_{(0)}$ is put equal to zero and that (7.632) for $P = 1, 2, \ldots, N-1$, and with $\alpha_{(0)}$ put equal to zero, follows as a kinematical consequence.

Now, turning to our dynamical equations (8.15), we find

$$(8.52)\qquad a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(P)} = -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(P,\,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(P,\,M)} - \frac{4}{v^2}V_r\lambda^r_{(P}\sum_{M=1}^{N-1}\alpha_{(M)}V_m\lambda^m_{(M)} + \frac{2\delta h}{v^2}V_r\lambda^r_{(P)},$$
$$(P = 1, 2, \ldots, N-1),$$

or, making use of (7.651) which are equally true in this case, we have

$$(8.521)\qquad a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(1)} = -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(1,\,M)} + \sum_{M=1}^{N-1}\alpha_{(M)}W_{(1,\,M)} - 4\alpha_{(1)}(\omega_{(1)})^2 - \frac{2\delta h}{v}\omega_{(1)},$$

$$(8.522)\qquad a_{rs}\hat{\hat{\eta}}^s\lambda^r_{(P)} = -v^2\sum_{M=1}^{N-1}\alpha_{(M)}K_{(P,\,M)} + \sum_{=1}^{-1}\alpha_{(M)}W_{(P,\,M)},$$
$$(P = 2, 3, \ldots, N-1).$$

† We have assumed that V_{rz} is not zero.

When we compare these equations with (7.632), we obtain precisely (7.662), (7.656) and (7.657). Thus we have the result:

THEOREM XLII (K) :—*In order that the motion of a holonomic conservative system may be stable in the kinematico-statical sense, it is necessary and sufficient that the values of* $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-1)}$ *as solutions of the differential equations* (7.662), (7.656) *and* (7.657) *should be permanently small for arbitrary values of the infinitesimal constant* δh.

In the case of steady motion the coefficients in these equations are constants.

The stability quadric being defined as in § 7.6, we have the analogues of Theorems XXXV and XXXVI as follows:

THEOREM XLIII (K) :—*In order that a steady motion with vanishing second curvature of a holonomic conservative system may be stable in the kinematico-statical sense, it is necessary and sufficient that the roots of the determinantal equation* (7.684) *should all be purely imaginary.*

THEOREM XLIV (K) :—*A steady motion of a holonomic conservative system, for which the second curvature of the trajectory vanishes and the first normal is a principal direction of the stability quadric, is stable in the kinematico-statical sense if and only if*

$$\text{(8.53)} \qquad \begin{cases} v^2 K_{(1,1)} + V_{mn}\lambda^m_{(1)}\lambda^n_{(1)} + 3v^2(\kappa_{(1)})^2 > 0, \\ v^2 K_{(P,P)} + V_{mn}\lambda^m_{(P)}\lambda^n_{(P)} > 0, \qquad (P = 2, 3, \ldots, N-1), \end{cases}$$

$K_{(P,P)}\,(P = 1, 2, \ldots, N-1)$ *being the Riemannian curvatures of the manifold of configurations corresponding to the two-space elements defined by the tangent to the trajectory and the other principal directions of the stability quadric,* $\lambda^r_{(1)}$ *being the unit vector in the direction of the first normal and* $\lambda^r_{(2)}, \lambda^r_{(3)}, \ldots, \lambda^r_{(N-1)}$ *being unit vectors in the directions of the remaining principal directions of the stability quadric.*

The periods of the stable vibrations are

$$\text{(8.531)} \qquad \begin{cases} 2\pi/(v^2 K_{(1,1)} + V_{mn}\lambda^m_{(1)}\lambda^n_{(1)} + 3v^2(\kappa_{(1)})^2)^{1/2}, \\ 2\pi/(v^2 K_{(P,P)} + V_{mn}\lambda^m_{(P)}\lambda^n_{(P)})^{1/2}, \qquad (P = 2, 3, \ldots, N-1). \end{cases}$$

CHAPTER IX.

STABILITY IN THE ACTION SENSE.

§ 9.1. *Equations for the components of the disturbance vector.*

When we come to consider stability in the action sense, we find equations very similar in form to those obtained in the last two chapters, but considerably simpler. It is important to remember that the metric here is the action line-element, and therefore the curvature tensor, covariant derivatives, etc., are not the same as in the last two chapters, where the kinematical line-element was employed. Most of the theory of the present chapter is really of a purely geometrical character, although presented in dynamical form for the purposes of this paper.

Let η^r be the infinitesimal vector drawn from a point P of the undisturbed curve C to the corresponding point P^* of the disturbed curve C^*. Then, if O and O^* are fixed corresponding points on the two curves, and if we write $OP = s$ and $O^*P^* = s^*$, the law of correspondence is $s = s^*$. Thus the equations of C^* (as geodesic) may be written

$$(9.11) \qquad q^{*r''} + \left\{ {m\,n \atop r} \right\}^* q^{*m'} q^{*n'} = 0,$$

where $q^{*r} = q^r + \eta^r$ and the accent denotes a derivative with respect to s. When we expand this expression and use the geodesic equations of C, we obtain without difficulty, just as in § 7.1,

$$(9.12) \qquad \overline{\overline{\eta}}^r + G^r_{msn}\eta^s q^{m'} q^{n'} = 0,$$

or, in covariant form,

$$(9.13) \qquad g_{rs}\overline{\overline{\eta}}^s + G_{rmsn} q^{m'} \eta^s q^{n'} = 0,$$

where g_{mn} is the fundamental tensor for the action line-element and G_{mnst} is the curvature tensor. This is *the tensorial equation for the components of the disturbance vector in the action sense.*

§ 9.2. *Equation for the magnitude of the disturbance vector.*

If, as in § 7.2, we write μ^r for the unit vector co-directional with the disturbance vector η^r, we see that, by using differentiation with respect to s instead of with respect to t, we may obtain the analogue of (7.24) in the form

$$(9.21) \qquad g_{rt}\overline{\overline{\eta}}^r \mu^t = \eta'' - \eta g_{rt}\overline{\mu}^r \overline{\mu}^t.$$

Then, multiplying (9.13) by μ^r and summing as indicated, we obtain

$$(9.22) \qquad \eta'' - \eta g_{rt}\overline{\mu}^r\overline{\mu}^t + G_{rmsn}\mu^r q^{m'} \eta^s q^{n'} = 0,$$

which may be written

$$(9.23) \qquad \eta'' + \eta\,(G_{mnst}\mu^m q^{n'} \mu^s q^{t'} - \overline{\mu}^2) = 0.$$

This is *the invariant equation for the magnitude of the disturbance in the action sense.*

We can without loss of generality choose the initial points O and O^* such that OO^* is perpendicular to C. Then, by the well-known property of geodesics, PP^* is always perpendicular to C, or, otherwise expressed, μ^r is perpendicular to $q^{r'}$. This we shall in future assume to be the case, so that

$$(9.24) \qquad g_{mn}\mu^m q^{n'} = 0.$$

Then $G_{mnst}\mu^m q^{n'} \mu^s q^{t'}$ is the Riemannian curvature of the manifold of configurations corresponding to the two-space element defined by μ^r and the direction of C. The following theorem results directly from (9.23):

THEOREM XLV (A) :—*If the Riemannian curvature of the manifold of configurations corresponding to every two-space element containing the direction of the curve of undisturbed motion is negative or zero at all points of the curve, then the motion is unstable in the action sense.*

§ 9.3. *The case of two degrees of freedom.*

Since the disturbance vector lies along the normal to the curve, we may put

(9.31) $$\mu^r = \varepsilon\nu^r, \qquad \varepsilon = \pm 1.$$

We have then

(9.311) $$\bar{\mu}^r = \varepsilon\bar{\nu}^r, \qquad \bar{\bar{\mu}} = \bar{\nu}.$$

But, since the curve is a geodesic, $\bar{\nu}^r$ is zero by (2.711) and therefore $\bar{\mu}$ is zero. Thus (9.23) may be written

(9.32) $$\eta'' + K\eta = 0,$$

where K is the Gaussian curvature of the manifold of configurations. Now η is by definition a positive quantity. Let us put $\beta = g_{mn}\eta^m\nu^n$, so that $\beta = \eta$ when ε is positive and $\beta = -\eta$ when ε is negative. The question of stability is therefore a question of the permanent smallness of β, where

(9.33) $$\beta'' + K\beta = 0.$$

Thus we have the result†:

THEOREM XLVI (A) :—*The motion of a holonomic conservative system with two degrees of freedom is stable in the action sense if the Gaussian curvature of the manifold of configurations is positive throughout the motion and unstable if it is negative or zero throughout the motion.*

Let us apply this theorem to the motion of a particle of unit mass on a plane, so that

(9.34) $$ds^2 = (h - V)(dx^2 + dy^2),$$

where h is the total energy, V the potential energy and (x, y) rectangular Cartesian co-ordinates. We find by calculation

(9.341) $$K = \frac{1}{2(h-V)^3}[(h-V)(V_{xx} + V_{yy}) + V_x^2 + V_y^2],$$

† This form of equation in connection with surface geodesics is, of course, well known. *Cf.* THOMSON and TAIT, *Natural Philosophy*, 1 (1879), 423.

where the subscripts denote partial derivatives. Thus, since $(h - V)$ is positive at all points of the undisturbed curve, we may state the result:

THEOREM XLVII:—*The motion of a particle of unit mass in a plane is stable in the action sense if the quantity*

$$(h - V)(V_{xx} + V_{yy}) + V_x^2 + V_y^2$$

is positive along the orbit and unstable if it is negative or zero along the orbit.

We note, in particular, that in the case of the logarithmic potential, for which $(V_{xx} + V_{yy})$ is zero, every motion is stable.

In the case of a particle of unit mass moving under the influence of a force directed towards a fixed point and varying inversely as the Pth power of the distance (where P is greater than unity), we find from (9.341)

(9.342) $$K = \frac{hk(1-P)}{2(h-V)^3 r^{P+1}},$$

where the attractive force is k/r^P and V vanishes at infinity. Thus we have the result:

THEOREM XLVIII:—*The motion of a particle in a plane under the influence of an attraction to a fixed point varying inversely as the Pth power of the (Euclidean) distance (P being greater than unity) is stable or unstable in the action sense according as the total energy is negative or positive, the potential energy being estimated in such a way as to vanish at infinity.*

Referring to equation (9.33), in the case of a steady motion K is constant along the undisturbed curve. Thus, if K is positive, the solution for the magnitude of the disturbance vector (counted positive when the disturbance lies to one side of the undisturbed curve and negative when it lies to the other side) is

(9.35) $$\beta = A \cos(sK^{\frac{1}{2}}) + B \sin(sK^{\frac{1}{2}}).$$

The periodic (action) distance is therefore

(9.351) $$s = 2\pi K^{-\frac{1}{2}},$$

and hence, by (4.12), the periodic time is

(9.352) $$t = \frac{\pi 2^{\frac{1}{2}}}{(h-V)K^{\frac{1}{2}}}.$$

§ 9.4. *The case of three degrees of freedom.*

Let λ^r be the unit vector tangent to the undisturbed curve C and let ν^r and ρ^r be two mutually perpendicular unit vectors perpendicular to λ^r and propagated parallelly along C. Let us write

(9.41) $$\eta^r = \beta\nu^r + \gamma\rho^r,$$

so that β and γ are the components of the disturbance vector along the directions of ν^r and ρ^r respectively, the component along the tangent being zero. Then, using the conditions of parallel propagation, we derive

(9.411) $$\overline{\overline{\eta}}^r = \beta''\nu^r + \gamma''\rho^r,$$

and hence

(9.412) $$g_{rs}\overline{\overline{\eta}}^r\nu^s = \beta'', \qquad g_{rs}\overline{\overline{\eta}}^r\rho^s = \gamma''.$$

But from (9.13) we find

(9.42) $$\begin{aligned} g_{rs}\overline{\overline{\eta}}^s\nu^r &= - G_{rmsn}\nu^r q^{m'}\eta^s q^{n'} \\ &= - \beta K_{(1,1)} - \gamma K_{(1,2)}, \end{aligned}$$

with the notation of § 6.7. Similarly

(9.421) $$g_{rs}\overline{\overline{\eta}}^s\rho^r = - \beta K_{(1,2)} - \gamma K_{(2,2)}.$$

By comparison with (9.412) we have the equations

(9.43) $$\begin{cases} \beta'' + \beta K_{(1,1)} + \gamma K_{(1,2)} = 0, \\ \gamma'' + \gamma K_{(2,2)} + \beta K_{(1,2)} = 0, \end{cases}$$

and, since

(9.44) $$\eta^2 = \beta^2 + \gamma^2,$$

we may state the result :

THEOREM XLIX (A) :—*In order that the motion of a holonomic conservative system with three degrees of freedom may be stable in the action sense, it is necessary and sufficient that the values of* β *and* γ *as solutions of the differential equations* (9.43) *should be permanently small.*

In the case of steady motion the quantities $K_{(1,1)}$, $K_{(1,2)}$ and $K_{(2,2)}$ are constant along the curve C, and we arrive at the following result directly from (9.43) :

THEOREM L (A) :—*In order that a steady motion of a holonomic conservative system with three degrees of freedom may be stable in the action sense, it is necessary and sufficient that the following conditions should be satisfied* :

(9.45) $$\begin{cases} K_{(1,1)} + K_{(2,2)} > 0, \\ K_{(1,1)}K_{(2,2)} \quad > [K_{(1,2)}]^2. \end{cases}$$

§ 9.5. *The general method of resolution along normals.*

Let $\lambda^r_{(0)}$ be the unit vector tangent to the undisturbed curve C and let $\lambda^r_{(1)}$, $\lambda^r_{(2)}$, ..., $\lambda^r_{(N-1)}$ be a system of mutually perpendicular unit vectors, perpendicular to $\lambda^r_{(0)}$ and propagated parallelly along C. We may write the disturbance vector in the form

(9.51) $$\eta^r = \alpha_{(1)}\lambda^r_{(1)} + \alpha_{(2)}\lambda^r_{(2)} + \ldots + \alpha_{(N-1)}\lambda^r_{(N-1)}.$$

By the method employed in § 9.4 we arrive at the equations

(9.52) $$\alpha''_P + \sum_{M=1}^{N-1} \alpha_{(M)}K_{(P,M)} = 0, \qquad (P = 1, 2, \ldots, N-1).$$

Since

(9.53) $$\eta^2 = (\alpha_{(1)})^2 + (\alpha_{(2)})^2 + \ldots + (\alpha_{(N-1)})^2,$$

we may state the result :

THEOREM LI (A) :—*In order that the motion of a holonomic conservative system may be stable in the action sense, it is necessary and sufficient that the values of* $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(N-1)}$ *as solutions of the differential equations* (9.52) *should be permanently small.*

In the case of steady motion the coefficients in these equations are constants.

For consideration of stability in the action sense we may define the *stability quadric* by the equation†

(9.54) $$G_{mnst}\xi^m\lambda^n_{(0)}\xi^s\lambda^t_{(0)} = constant.$$

The condition that the directions $\lambda^r_{(P)}$ and $\lambda^r_{(Q)}$ should be conjugate is

(9.541) $$K_{(P,\,Q)} \equiv G_{mnst}\lambda^m_{(P)}\lambda^n_{(0)}\lambda^s_{(Q)}\lambda^t_{(0)} = 0.$$

In the case of steady motion (to which we shall now confine ourselves) the vectors $\lambda^r_{(P)}$ $(P = 1, 2, \ldots, N-1)$ may be chosen in the principal directions of the stability quadric, the tangential direction $\lambda^r_{(0)}$ being obviously a principal direction. When the normal vectors have been so chosen, equations (9.52) become

(9.55) $$\alpha''_{(P)} + \alpha_{(P)}K_{(P,\,P)} = 0, \qquad (P = 1, 2, \ldots, N-1),$$

and we see that there is stability if, and only if, $K_{(P,\,P)}$ is positive for $P = 1, 2, \ldots, N-1$. But the Riemannian curvature of the manifold of configurations corresponding to any two-space element containing the tangential direction may be written

(9.56) $$K = G_{mnst}\xi^m\lambda^n_{(0)}\xi^s\lambda^t_{(0)},$$

where

(9.57) $$\xi^r = \beta_{(1)}\lambda^r_{(1)} + \beta_{(2)}\lambda^r_{(2)} + \ldots + \beta_{(N-1)}\lambda^r_{(N-1)},$$

and

(9.571) $$(\beta_{(1)})^2 + (\beta_{(2)})^2 + \ldots + (\beta_{(N-1)})^2 = 1.$$

Hence, substituting in (9.56), we have

(9.58) $$K = (\beta_{(1)})^2K_{(1,\,1)} + (\beta_{(2)})^2K_{(2,\,2)} + \ldots + (\beta_{(N-1)})^2K_{(N-1,\,N-1)}.$$

Thus we may state the result :

THEOREM LII (A) :—*A steady motion of a holonomic conservative system is stable in the action sense if, and only if, the Riemannian curvature of the manifold of configurations corresponding to every two-space element containing the tangent to the undisturbed curve is positive.*

† *Cf.* Equation (7.691).